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2008 J. Phys. A: Math. Theor. 41 275209

(http://iopscience.iop.org/1751-8121/41/27/275209)

J. Phys. A: Math. Theor. **41** (2008) 275209 (33pp) doi:10.1088/1751-8113/41/27/275209

Classification of the line-soliton solutions of KPII

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Received 24 April 2008
Published 16 June 2008
Online at stacks.iop.org/JPhysA/41/275209

Abstract
In the previous papers (notably, Kodama Y 2004 *J. Phys. A: Math. Gen.* **37** 11169–90, Biondini G and Chakravarty S 2006 *J. Math. Phys.* **47** 033514), a large variety of line-soliton solutions of the Kadomtsev–Petviashvili II (KPII) equation was found. The line-soliton solutions are solitary waves which decay exponentially in the (x, y)-plane except along certain rays. In this paper, it is shown that those solutions are classified by asymptotic information of the solution as $|y| \to \infty$. The present work then unravels some interesting relations between the line-soliton classification scheme and classical results in the theory of permutations.

PACS numbers: 02.30.Ik, 02.10.Ox

1. The KPII equation and its line-soliton solutions

The Kadomtsev–Petviashvili (KP) equation

$$\frac{\partial}{\partial x}\left(-4\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+6u\frac{\partial u}{\partial x}\right)+3\sigma^2\frac{\partial^2 u}{\partial y^2}=0, \tag{1.1}$$

where $u = u(x, y, t)$ and $\sigma^2 = \pm 1$, describes the evolution of small-amplitude, quasi-two-dimensional solitary waves in a weakly dispersive medium [13]. The case $\sigma^2 = -1$ corresponding to positive dispersion is known as the KPI equation, whereas the negative dispersion ($\sigma^2 = 1$) case is referred to as the KPII equation. The KP equation arises in many physical applications including water waves and plasmas (see, e.g. [12] for a review). It is a completely integrable system with remarkably rich mathematical structure which is well documented in several monographs [1, 11, 16, 19, 21]. Particularly, it has been known that the solutions of the KP equation can be expressed in terms of the τ-function [11, 26],

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2}\log\tau(x, y, t). \tag{1.2}$$

In this paper, we consider a class of solutions whose τ-function is given by the Wronskian determinant [10, 26]

$$\tau(x,y,t) = \mathrm{Wr}(f_1,\ldots,f_N) = \begin{pmatrix} f_1 & f_2 & \cdots & f_N \\ f_1' & f_2' & \cdots & f_N' \\ \vdots & \vdots & & \vdots \\ f_N^{(N-1)} & f_2^{(N-1)} & \cdots & f_N^{(N-1)} \end{pmatrix}, \tag{1.3}$$

with $f^{(i)} = \partial^i f/\partial x^i$, and where the functions $\{f_n\}_{n=1}^N$ are a set of linearly independent solutions of the linear system

$$\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2}, \qquad \frac{\partial f}{\partial t} = \frac{\partial^3 f}{\partial x^3}. \tag{1.4}$$

In particular, we investigate the *line-soliton* solutions of the KPII equation, which are real, non-singular solutions localized along certain directions in the (x, y)-plane, and decay exponentially everywhere else. For example, a one-soliton solution is obtained by choosing $N = 1$ in equation (1.3), and $\tau(x, y, t) = f(x, y, t) = \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}$, where

$$\theta_m(x,y,t) = k_m x + k_m^2 y + k_m^3 t + \theta_{m,0} \tag{1.5}$$

with real parameters $\theta_{m,0}, k_m$ for $m = 1, 2$, and $k_1 < k_2$. The above choices yield the traveling-wave solution

$$u(x,y,t) = \tfrac{1}{2}(k_2 - k_1)^2 \operatorname{sech}^2 \tfrac{1}{2}(\theta_2 - \theta_1) = \Phi(\mathbf{k}\cdot\mathbf{r} + \omega t), \tag{1.6}$$

where $\mathbf{r} = (x, y)$. The wave vector $\mathbf{k} := (l_x, l_y) = \left(k_1 - k_2, k_1^2 - k_2^2\right)$ and the frequency ω satisfy the dispersion relation,

$$-4\omega l_x + l_x^4 + 3l_y^2 = 0. \tag{1.7}$$

The solitary wave given by equation (1.6) is localized in the (x, y)-plane along the line $L : \theta_1 = \theta_2$ whose normal has the slope $c = l_y/l_x = k_1 + k_2$. The one-soliton solution is characterized by two physical parameters, namely, the *soliton amplitude* $a = k_2 - k_1$ and the *soliton direction* $c = k_1 + k_2$. The soliton direction can also be expressed as $c = \tan\alpha$, where α is the angle, measured counterclockwise, between the line L and the positive y-axis. Conversely, any given choice of the amplitude ($a > 0$) and direction of the soliton gives the phase parameters k_1 and k_2 uniquely as $k_1 = \frac{1}{2}(c - a)$ and $k_2 = \frac{1}{2}(c + a)$. Note that when $c = 0$ (equivalently, $k_1 = -k_2$) the solution in equation (1.6) becomes y-independent and reduces to the one-soliton solution of the Korteweg–de Vries (KdV) equation.

General line-soliton solutions. Like KdV, the KPII equation admits multi-soliton solutions which can also be constructed via the Wronskian formulation of equation (1.3) by choosing M phases $\{\theta_m\}_{m=1}^M$ defined as in equation (1.5) with distinct real *phase parameters* $k_1 < k_2 < \cdots < k_M$ and then defining the functions

$$f_n(x,y,t) = \sum_{m=1}^{M} a_{nm}\mathrm{e}^{\theta_m}, \qquad n = 1, 2, \ldots, N, \tag{1.8}$$

which give finite dimensional solutions of equations (1.4). The constant coefficients a_{nm} define the $N \times M$ *coefficient matrix* $A := (a_{nm})$, all of whose $N \times N$ minors must be non-negative to ensure that the τ-function $\tau(x, y, t)$ has no zeros in the (x, y)-plane for all t, so that the corresponding KPII solution $u(x, y, t)$ resulting from equation (1.2) is non-singular.

However, the multi-soliton solution space of the KPII equation turns out to be much richer than that of the (1+1)-dimensional KdV equation due to the dependence of the KPII

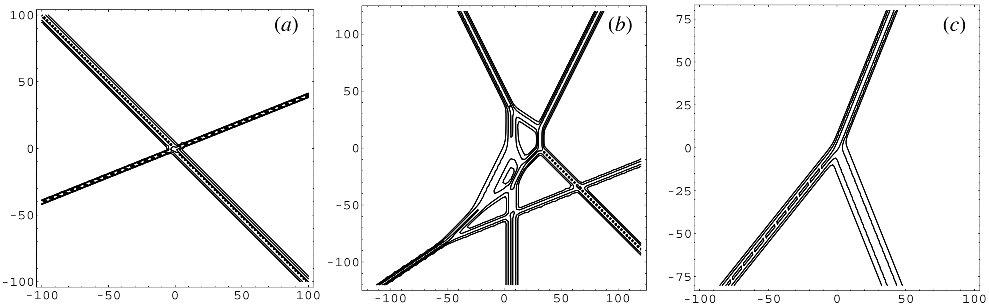

Figure 1. Line-soliton solutions of the KPII equation illustrating different interaction patterns: (*a*) a two-soliton solution, (*b*) a partially resonant (3, 3)-soliton and (*c*) a Miles resonance (Y-junction). Here and in all following figures, the horizontal and vertical axes are, respectively, x and y, and the graphs show contour lines of the solution $u(x, y, t) = 2\partial_x^2 \log \tau(x, y, t)$ for fixed t.

solutions on the additional spatial variable y. Asymptotically, as $y \to \pm\infty$, there exist certain (non-decaying) directions which are invariant in t, and along which the solution has the form of a plane wave similar to the one-soliton solution in equation (1.6). These asymptotic solitary wave structures, referred to as *asymptotic* line solitons in [2], have varying amplitudes and directions depending on M, N and the values of the phase parameters $k_1, \ldots, k_M$. More significantly, the number N_- of asymptotic line solitons as $y \to -\infty$ is in general different from the number N_+ of the asymptotic line solitons as $y \to \infty$, with $N_- = M - N$ and $N_+ = N$. Such multi-soliton configurations derived from equation (1.8) are called (N_-, N_+)-soliton solutions of KPII [2, 4]. For example, figure 1(*c*) exhibits a (2, 1)-soliton solution, also known as the Miles resonance solution [18]. At the interaction vertex or Y-junction, the three interacting line solitons with wave numbers $\mathbf{k}_a$ and frequencies ω_a $(a = 1, 2, 3)$ satisfy the fundamental three-wave resonance condition

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3, \qquad \omega_1 + \omega_2 = \omega_3. \tag{1.9}$$

The (N_-, N_+)-soliton solutions exhibit a variety of time-dependent spatial interaction patterns including the formation of intermediate line solitons in the (x, y)-plane [4, 15, 17]. In contrast to these nontrivial interactions exhibited by the KPII solitons, the KdV multi-soliton solutions experience only a phase shift after collision.

N-soliton solutions. When $N_- = N_+ = N$ (i.e., when $M = 2N$), the corresponding solutions consist of the same number of asymptotic line solitons as $y \to \pm\infty$. If in addition, the direction and amplitude of each of the N line solitons as $y \to -\infty$ are *pairwise* equal to each of the N line solitons as $y \to \infty$, then the corresponding solutions are simply referred to as the N-soliton solutions of the KPII equation. It will be evident from the discussions in the following sections that each N-soliton solution can be regarded as a configuration of N interacting asymptotic line solitons where the amplitude and direction of the nth line soliton are given by

$$a_n = k_{j_n} - k_{i_n}, \qquad c_n = k_{i_n} + k_{j_n}, \qquad n = 1, \ldots, N. \tag{1.10}$$

Thus, the nth line soliton is parametrized by a pair $\left(k_{i_n}, k_{j_n}\right)$ of distinct phase parameters with $1 \leqslant i_n < j_n \leqslant 2N$, or equivalently, by the index pair $[i_n, j_n]$.

The Y-junction solution found by Miles [18] is perhaps the earliest evidence of resonant structure present in the line-soliton solutions of KPII. Subsequently, this solution was reconstructed using different algebraic methods in several earlier works (see, e.g. [9, 20, 22]), and more general types of resonant line-soliton solutions of KPII were reported in some recent works including [4, 17, 23]. The properties of the general line-soliton solutions were systematically investigated in [2] where these solutions were characterized by developing an asymptotic analysis of the τ-function. The N-soliton solutions were extensively studied in [15]. In particular, an explicit characterization of the N-soliton solutions space in terms of Grassmannian $Gr(N, 2N)$ was presented for the first time in [15].

In this paper, we extend the work in [2] by providing a combinatorial description of the general line-soliton solutions of KPII. Furthermore, we show that the class of the general line-soliton solutions can be enumerated employing these combinatorial properties. The paper is planned as follows. In section 2, we first review the asymptotic properties of the τ-function, the asymptotic line solitons and their characterizations by distinct pairs of phase parameters for the general $(N, M-N)$-solitons of KPII. Then we establish a one-to-one correspondence between the $(N, M-N)$-soliton equivalence classes and certain type of permutations of the index set $\{1, 2, \ldots, M\}$ called derangements (proposition 2.9 and definition 2.10). We also make some remarks on those equivalence classes of solutions in terms of the positive Grassmann cells, and find a generating function (polynomial) for $(N, M-N)$-soliton solutions (proposition 2.11). By way of an example of our general result, we give a comprehensive classification of the $(2, 2)$-soliton solutions in section 3. In section 4.1, we show that the line-soliton solution space of the KPII equation is divided into dual sub-classes of $(N, M-N)$- and $(M-N, N)$-soliton solutions under the action of spacetime inversion $(x, y, t) \to (-x, -y, -t)$. We then give a combinatorial interpretation of this discrete symmetry of the KPII equation. Next in section 4.2, we consider the N-soliton solutions, and in section 4.3, we describe how to construct such solutions starting only from the physical data set of N amplitudes and N directions of the associated line solitons. We also show that there exists a one-to-one correspondence between N-soliton solution space and the set of all fixed-point free involutions of the permutation group of $2N$ elements. Finally, in section 4.4, we exhibit how the combinatorial properties of the N-soliton solutions provide a further refinement of the N-soliton solution space (proposition 4.10), and thus recover the results of [15].

2. The KPII τ-function and asymptotic line solitons

In this section, we investigate the general properties and asymptotic behavior of the τ-function associated with the general line-soliton solutions of the KPII equation. As before, we consider the Wronskian form of the τ-function given by equation (1.3), where the functions $\{f_n\}_{n=1}^N$ are linear combinations of exponentials with M distinct phases as in equation (1.8). Furthermore, we can assume without loss of generality that the phase parameters are ordered as $k_1 < k_2 < \cdots < k_M$.

2.1. Properties of the τ-function

The Wronskian in equation (1.3) can be expressed as

$$\tau(x, y, t) = \det(A\Theta K), \tag{2.1}$$

where $A = (a_{nm})$ is the $N \times M$ coefficient matrix, $\Theta = \operatorname{diag}(e^{\theta_1}, \ldots, e^{\theta_M})$ and the $M \times N$ matrix K is given by $K = \left(k_m^{n-1}\right), m = 1, 2, \ldots, M, n = 1, 2, \ldots, N$. Expanding the determinant in equation (2.1) using the Binet–Cauchy formula yields the following explicit form of the τ-function:

$$\tau(x, y, t) = \sum_{1 \leqslant m_1 < \ldots < m_N \leqslant M} A(m_1, \ldots, m_N) \exp[\theta(m_1, \ldots, m_N)] \prod_{1 \leqslant s < r \leqslant N} \left(k_{m_r} - k_{m_s}\right), \tag{2.2}$$

where the phase combination is defined by

$$\theta(m_1, \ldots, m_N) := \theta_{m_1} + \theta_{m_2} + \cdots + \theta_{m_N},$$

$A(m_1, \ldots, m_N)$ is the $N \times N$ minor of A obtained from columns $1 \leqslant m_1 < \cdots < m_N \leqslant M$, and the product term in equation (2.2) is the Van der Monde determinant obtained using the rows $1 \leqslant m_1 < \cdots < m_N \leqslant M$ of the matrix K in equation (2.1). The basic properties of the τ-function following from equation (2.2) are listed below.

Property 2.1.

(i) The τ-function is a linear combination of real exponentials, where each exponential term contains combinations of N out of M distinct phases given by $\theta(m_1, \ldots, m_N)$. Any given phase combination $\theta(m_1, \ldots, m_N)$ actually appears in the τ-function if and only if the corresponding minor $A(m_1, \ldots, m_N)$ is nonzero. Thus, there are *at most* $\binom{M}{N}$ terms in the τ-function.

(ii) If $M = N$, the corresponding τ-function in equation (2.2) contains only one exponential term which generates the trivial solution $u(x, y, t) = 0$ of KPII via equation (1.2). Hence, $M > N$ for nontrivial solutions.

(iii) If $\mathrm{rank}(A) < N$, then all the $N \times N$ minors of A vanish identically, leading to the trivial case $\tau = 0$. Moreover, for $\mathrm{rank}(A) = N$, if all minors $A(m_1, \ldots, m_N) \geqslant 0$ then $\tau(x, y, t) > 0, \forall (x, y, t) \in \mathbb{R}^3$. Therefore, the resulting solution $u(x, y, t)$ of the KPII equation is non-singular.

(iv) The transformation $A \to CA$ where $C \in GL(N, \mathbb{R})$ corresponds to an overall rescaling $\tau \to \det(C)\tau$, of the τ-function in equation (2.1), which leaves the solution $u(x, y, t)$ invariant. This $GL(N, \mathbb{R})$ freedom can be exploited to choose the coefficient matrix A in equation (1.8) to be in the reduced row-echelon form (RREF).

(v) The transformation $A \to AD, \Theta \to D^{-1}\Theta$ where $D \in GL(M, \mathbb{R})$ leaves the τ-function in equation (2.1) invariant. In particular, a diagonal matrix D with diagonal elements $d_m > 0, m = 1, \ldots, M$, leaves the functions $\{f_n\}_1^N$ in equation (1.8) invariant by simultaneously rescaling the mth column of A by d_m, and shifting each phase constant in equation (1.5) as $\theta_{m,0} \to \theta_{m,0} - \log(d_m)$.

(vi) If for any given n, m in equation (1.8), we take $f_n = \mathrm{e}^{\theta_m}$ such that the nth row of A has only one nonzero entry $a_{nm} = 1$, then the minors $A(m_1, \ldots, m_N) = 0, m \notin \{m_1, \ldots, m_N\}$. The resulting τ-function can be expressed as $\tau(x, y, t) = \mathrm{e}^{\theta_m}\tau_0(x, y, t)$, where $\tau_0(x, y, t)$ contains at most $M - 1$ phases (all but θ_m) which appear in phase combinations of $N - 1$ distinct phases. It is then evident from equation (1.2) that $\tau(x, y, t)$ and $\tau_0(x, y, t)$ generate the same solution of KPII. Such τ-functions are *reducible* in the sense that they can be effectively obtained from a Wronskian of $N - 1$ functions with $M - 1$ distinct phases.

For the remainder of this paper, we will consider the coefficient matrix A to be in RREF. Furthermore, to avoid trivial and reducible cases, and to ensure that the solution $u(x, y, t)$ of KPII resulting from the τ-function in equation (2.2) are non-singular, we will impose the following restrictions on the coefficient matrix A.

Condition 2.2.

(1) Positivity: $\mathrm{rank}(A) = N < M$ and all nonzero minors of A are positive.

(2) Irreducibility: each column of A contains at least one nonzero element, and each row of A contains at least one nonzero element in addition to the pivot (first nonzero) entry.

Remark 2.3. The matrices satisfying condition 2.2(i) are called totally non-negative (TNN) matrices. The classification of the (N_-, N_+)-soliton solutions is then given by the classification of the $N \times M$ irreducible TNN matrices A in RREF. From a more geometric perspective, each TNN matrix parametrizes a unique cell in the TNN Grassmannian $Gr^+(N, M)$ (see, e.g. [24]), and the classification of the soliton solutions corresponds to a further refinement of the Schubert decomposition of $Gr(N, M)$ into TNN Grassmann cells (see [15] for the case $M = 2N$). The refinement is given by a classification of the coefficient matrix A, and the minors $A(m_1, \ldots, m_N)$ represent the Plücker coordinates of $Gr(N, M)$. We will discuss the geometric structure of this classification in a future communication [7].

2.2. Dominant phase combinations and asymptotic line solitons

The spatial structure of the solution $u(x, y, t)$ is determined from the asymptotic behavior of the τ-function in the (x, y)-plane and for finite values of t. Note that the τ-function in equation (2.2) associated with an irreducible coefficient matrix A is a sum of real exponentials with positive coefficients. If only one phase combination $\theta(m_1, \ldots, m_N)(x, y, t)$ in the τ-function is dominant in a certain region of the (x, y)-plane at a given time, then $\tau \sim \exp(\theta(m_1, \ldots, m_N))$. Consequently, the solution $u(x, y, t)$ of KPII generated by the τ-function (2.2) is exponentially small at all points in the interior of any dominant region, and is localized at the boundaries where a balance exists between at least two dominant phase combinations in the τ-function (2.2). Such boundary is identified by the equation $\theta(m_1, \ldots, m_N) = \theta(m'_1, \ldots, m'_N)$, which defines a line segment in the (x, y)-plane for each t. Note that this phenomenon also arises for the one-soliton solution (1.6), which is localized along the line $\theta_1 = \theta_2$ corresponding to the boundary of the two regions of the (x, y)-plane where θ_1 and θ_2 dominate. In the one-soliton case, these two regions are simply half-planes, whereas in the general case the dominant regions could be bounded or unbounded (see, e.g. figure 2). A detailed analysis of the asymptotic behavior of the τ-function was carried out in [2], the main results are summarized below.

Proposition 2.4. *The asymptotic properties of τ-function (2.2) for finite values of t, and for generic values of phase parameters $k_1, \ldots, k_M$, are as follows:*

(i) The dominant phase combinations of the τ-function in adjacent regions of the (x, y)-plane as $y \to \pm\infty$, contain $N - 1$ common phases and differ by only a single phase. The transition $\theta(i, m_2, \ldots, m_N) \mapsto \theta(j, m_2, \ldots, m_N)$ between any two such dominant phase combinations occurs along the line defined by $[i, j] : \theta_i = \theta_j, i \neq j$, where a single phase θ_i in one dominant phase combination is replaced by a phase θ_j.

(ii) Along the single-phase transition line $[i, j]$, the dominant phase balance yields

$$\tau(x, y, t) \sim C_i \mathrm{e}^{\theta(i,m_2,\ldots,m_N)} + C_j \mathrm{e}^{\theta(j,m_2,\ldots,m_N)} \tag{2.3a}$$

asymptotically as $y \to \infty$ or as $y \to -\infty$, where the coefficients C_i, C_j depend on appropriate Van der Monde determinants and non-vanishing minors of the coefficient matrix A. The asymptotic behavior of the solution along $[i, j]$ is given by

$$u(x, y, t) \sim \tfrac{1}{2}(k_j - k_i)^2 \operatorname{sech}^2 \tfrac{1}{2}(\theta_j - \theta_i + \delta_{ij}), \tag{2.3b}$$

which defines an asymptotic line soliton.

(iii) The number of the asymptotic line solitons is invariant in time, and so are their amplitudes and directions. In particular, the soliton direction is given by the normal direction of $[i, j]$, which is $c_{i,j} = k_i + k_j$, and the soliton amplitude is given by $a_{ij} = |k_i - k_j|$.

In view of proposition 2.4, it is natural to denote an asymptotic line soliton by the index pair $[i, j]$ which labels the asymptotic direction as well as the phase parameters in equation (2.3*b*). The asymptotic properties of the τ-function reveal that there exists a certain pairing (k_i, k_j), $1 \leqslant i < j \leqslant M$ between the phase parameters, which in turn defines an asymptotic line soliton. However, proposition 2.4 does not specify how to determine these pairings in a given solution, or equivalently, which phase combinations are actually dominant in a given τ-function as $|y| \to \infty$. It is still necessary to identify the particular set of asymptotic line solitons associated with any given τ-function of equation (2.2). For this purpose, we first need a result derived in [2][lemma 3.1] regarding the dominant phases.

Lemma 2.5 (Dominant phase conditions)**.** *Along the line $[i, j] : \theta_i = \theta_j$ with $i < j$, the phases $\theta_1, \ldots, \theta_M$ satisfy the following relations, where $\theta := \theta_i = \theta_j$:*

(i) as $y \to \infty, \theta_m < \theta, \forall m \in \{i+1, \ldots, j-1\}$, and $\theta_m > \theta, \forall m \in \{1, \ldots, i-1, j+1, \ldots, M\}$;

(ii) as $y \to -\infty, \theta_m > \theta, \forall m \in \{i+1, \ldots, j-1\}$, and $\theta_m < \theta, \forall m \in \{1, \ldots, i-1, j+1, \ldots, M\}$.

The proof of this lemma based on the ordering $k_1 < \cdots < k_M$ and equation (1.5) is straightforward. Lemma 2.5 provides a simple yet useful way to determine the dominant phase combinations along the line $[i, j]$. However, a given phase combination $\theta(m_1, \ldots, m_N)$ can only be dominant if it is in fact *present* in the τ-function of equation (2.2), i.e., if the corresponding coefficient minor $A(m_1, \ldots, m_N) \neq 0$ in the τ-function. Therefore, in order to obtain a complete characterization of the asymptotic line solitons, it is necessary to consider the structure of the $N \times M$ coefficient matrix A in addition to lemma 2.5. Each asymptotic line-soliton $[i, j]$ of a (N_-, N_+)-soliton solution of KPII is uniquely determined by a pair of columns of the coefficient matrix A, as prescribed below. Once again, the details can be found in [2].

Proposition 2.6. *The (N_-, N_+)-soliton solution of KPII generated from the τ-function in equation (2.2) has exactly $N_+ = N$ asymptotic line solitons as $y \to \infty$ and $N_- = M - N$ asymptotic line solitons as $y \to -\infty$. The necessary and sufficient conditions for an index pair $[i, j]$ to identify an asymptotic line soliton are determined by the ranks of two sub-matrices of A defined below in terms of their column indices*

$$X[ij] := [1, 2, \ldots, i-1, j+1, \ldots, M] \qquad Y[ij] := [i+1, \ldots j-1].$$

The rank conditions are then as follows:

(i) Each asymptotic line soliton as $y \to \infty$ is labeled by a unique index pair $[e_n, j_n]$ with $e_n < j_n$ where $\{e_n\}_{n=1}^N$ label the pivot columns of A. Moreover, if $\operatorname{rank}(X[e_n j_n]) := r_n$, then

$$r_n \leqslant N - 1 \qquad \textit{and}$$
$$\operatorname{rank}(X[e_n j_n]|e_n) = \operatorname{rank}(X[e_n j_n]|j_n) = \operatorname{rank}(X[e_n j_n]|e_n, j_n) = r_n + 1.$$

(ii) An asymptotic line soliton as $y \to -\infty$ is labeled by a unique index pair $[i_n, g_n]$ with $i_n < g_n$ where $\{g_n\}_{n=1}^{M-N}$ label the non-pivot columns of A. Moreover, if $\operatorname{rank}(Y[i_n g_n]) := s_n$, then

$$s_n \leqslant N - 1 \qquad \textit{and}$$
$$\operatorname{rank}(Y[i_n g_n]|i_n) = \operatorname{rank}(Y[i_n g_n]|g_n) = \operatorname{rank}(Y[i_n g_n]|i_n, g_n) = s_n + 1.$$

Above, $(Z|m, n)$ denotes the sub-matrix Z of A augmented by the columns m and n of A.

Given the τ-function data, which consist of M distinct phase parameters $k_1, \ldots, k_M$ and a matrix A satisfying condition 2.2, propositions 2.4 and 2.6 provide an explicit way to identify all the asymptotic line solitons of the corresponding solution of the KPII equation. This method is illustrated via the examples below.

Example 2.7. Figure 2(a) illustrates a (2, 1)-soliton Y-junction solution [18] describing the resonant interaction of two line solitons mentioned in section 1 (see figure 1(c)). This solution corresponds to $N = 1$, $M = 3$, and is generated by the τ-function and the coefficient matrix A,

$$\tau(x, y, t) = \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3}, \qquad A = (1 \quad 1 \quad 1).$$

In this case we know from proposition 2.6 that the number of asymptotic line solitons as $y \to \infty$ and as $y \to -\infty$ are one and two, respectively. Applying the rank conditions from proposition 2.6(i) to the pivot column $e_1 = 1$, we see that for the soliton $[1, j_1]$ as $y \to \infty$, $\mathrm{rank}(X[1 j_1]) = 0$ since $N = 1$. Hence, $j_1 = 3$, so that the line soliton is [1, 3], which (according to proposition 2.4) corresponds to the dominant balance of the phases θ_1 and θ_3 in the τ-function. As $y \to -\infty$, proposition 2.6(ii) implies that for the non-pivot columns $g_1 = 2$ and $g_2 = 3$ we should have $\mathrm{rank}(Y[i_1 2]) = \mathrm{rank}(Y[i_2 3]) = 0$ as well. Consequently, $i_1 = 1, i_2 = 2$, and the resulting line solitons [1, 2], [2, 3] correspond to the dominant balance of the phase pairs (θ_1, θ_2) and (θ_2, θ_3), respectively. Thus, the (x, y)-plane is partitioned in three disjoint regions where each of the phases θ_1, θ_2 and θ_3 dominates, and the solution is localized along the phase transition lines which mark the asymptotic line solitons.

Example 2.8. Consider $N = 2$ and $M = 4$ corresponding to a (2, 2)-soliton solution as shown in figure 2(b), and generated by the τ-function in equation (1.3) with

$$f_1 = \mathrm{e}^{\theta_1} - \mathrm{e}^{\theta_4}, \qquad f_2 = \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} + \mathrm{e}^{\theta_4}, \qquad A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

The pivot columns of A are labeled by the indices $\{e_1, e_2\} = \{1, 2\}$, and the non-pivot columns by the indices $\{g_1, g_2\} = \{3, 4\}$. According to proposition 2.6, the number of asymptotic line solitons are $N_+ = N_- = 2$. They are identified by the index pairs $[1, j_1]$, $[2, j_2]$ as $y \to \infty$, for some $j_1 > 1$ and $j_2 > 2$; and by the index pairs $[i_1, 3]$, $[i_2, 4]$ as $y \to -\infty$, for some $i_1 < 3$ and $i_2 < 4$. We first determine the asymptotic line solitons as $y \to \infty$ using the rank conditions prescribed in proposition 2.6(i). For the first pivot column $e_1 = 1$, starting from $j = 2$ and then incrementing the value of j by one, we check the rank of each sub-matrix $X[1j]$. Proceeding in this way, we find that the rank conditions are satisfied when $j = 3$: $X[13] = \binom{-1}{1}$. So $\mathrm{rank}(X[13]) = 1 = N - 1$. Moreover, $\mathrm{rank}(X[13]|1) = \mathrm{rank}(X[13]|3) = \mathrm{rank}(X[13]|1, 3) = 2$. Thus, the first asymptotic line soliton as $y \to \infty$ is identified by the index pair [1, 3]. For $e_2 = 2$, proceeding in a similar manner as above we find that $j = 3$ does not satisfy the rank conditions (since $X[23]$ has rank 2) but $j = 4$ does. Therefore, the asymptotic line solitons as $y \to \infty$ are given by the index pairs [1, 3] and [2, 4].

Then we consider the asymptotics for $y \to -\infty$. Starting with the non-pivot column $g_1 = 3$, we apply the rank conditions in proposition 2.6(ii) to the column $i = 2$. Then, we have $Y[23] = \emptyset$, and $\mathrm{rank}(Y[23]|2) = \mathrm{rank}(Y[23]|3) = \mathrm{rank}(Y[23]|2, 3) = 1$. Hence, the pair [2, 3] identifies an asymptotic line soliton as $y \to -\infty$. For $g_2 = 4$, we consider $i = 1, 2, 3$ and find that the rank conditions are satisfied only for $i = 1$. In this case, $Y[14] = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, so $\mathrm{rank}(Y[14]) = 1 = N - 1$ and $\mathrm{rank}(Y[14]|1) = \mathrm{rank}(Y[14]|4) = \mathrm{rank}(Y[24]|1, 4) = 2$. Thus, the index pair [1, 4] identifies the other asymptotic line soliton as $y \to -\infty$.

The dominant phase regions in the (x, y)-plane for this (2, 2)-soliton solution can also be identified from proposition 2.4. First note that the dominant phase combination along $x \to -\infty$ is given by $\theta(1,2) = \theta_1 + \theta_2$ for finite y. This follows from equation (1.5) and the ordering $k_1 < k_2 < k_3 < k_4$ of the phase parameters. As the slope of the normal to the transition line $\theta_i = \theta_j$, given by the direction $c_{ij} = k_i + k_j$, decreases from the negative x-axis to the positive x-axis for $y > 0$, the asymptotic line solitons are sorted clockwise as [2, 4] and [1, 3] as $y \to \infty$. Thus, the dominant phase combinations associated with single-phase transitions (cf proposition 2.4) as $y \to \infty$ are given by

$$\theta(1,2) \xrightarrow{2\to 4} \theta(1,4) \xrightarrow{1\to 3} \theta(3,4).$$

When $y < 0$, the soliton direction parameter c_{ij} increases from the negative x-axis to the positive x-axis. Consequently, the asymptotic line solitons as $y \to -\infty$ are sorted counterclockwise as [2, 3] and [1, 4], and determine the dominant phase combinations in the (x, y)-plane for $y \to -\infty$ as follows:

$$\theta(1,2) \xrightarrow{2\to 3} \theta(1,3) \xrightarrow{1\to 4} \theta(3,4).$$

The asymptotic line solitons and dominant phase combinations are shown in figure 2 where the phase parameters are chosen such that $c_{14} = 1 > c_{23} = 0$. Note that in addition to the unbounded dominant regions there is also a bounded region in the xy-plane where $\theta(2,4)$ is the dominant phase combination. The boundaries of this region are formed by the line solitons [1, 4] and [2, 3] as $y \to -\infty$, together with the intermediate line-soliton [1, 2].

The above examples illustrate how to identify the asymptotic line solitons and the dominant phase combinations of a (N_-, N_+)-soliton solution of KPII from the τ-function in an algorithmic fashion. First, apply the rank conditions given in proposition 2.6 to each pivot column and to each non-pivot column of the coefficient matrix A to identify the asymptotic line solitons as $y \to \pm\infty$. Next, note that for a τ-function in equation (2.2) associated with a coefficient matrix A in RREF, the dominant phase combination as $x \to -\infty$ is uniquely given by $\theta(e_1, \ldots, e_N)$. This is due to the fact that the phase parameters are ordered as $k_1 < k_2 < \cdots < k_M$, and the coefficient $A(e_1, \ldots, e_N)$ of the term $\mathrm{e}^{\theta(e_1,\ldots,e_N)}$ being the minor of pivot columns, is lexicographically the first non-vanishing minor of A with $A(e_1, \ldots, e_N) = 1$. Since the line solitons are sorted according to their direction parameter c_{ij}, the dominant phase combinations in the (x, y)-plane can then be determined from proposition 2.4 starting from the dominant phase combination $\theta(e_1, \ldots, e_N)$ as $x \to -\infty$.

2.3. Index pairing and derangements

We show here that the set of unique index pairings in proposition 2.6, $\{[e_n, j_n]\}_{n=1}^{N} \cup \{[i_n, g_n]\}_{n=1}^{M-N}$ identifying the asymptotic line solitons as $|y| \to \infty$, has a combinatorial interpretation. Let $[M] := \{1, 2, \ldots, M\}$ be the integer set and recall that $\{e_1, \ldots, e_N\} \cup \{g_1, \ldots, g_{M-N}\}$ is a disjoint partition of $[M]$. Define the pairing map $\pi : [M] \to [M]$ according to proposition 2.6(i) and (ii) as

$$\pi(e_n) = j_n, n = 1, 2, \ldots, N, \qquad \pi(g_n) = i_n, n = 1, 2, \ldots, M - N, \tag{2.4}$$

where e_n and g_n are, respectively, the pivot and non-pivot indices of the coefficient matrix A. Then one can show that the map $\pi : [M] \to [M]$ is a bijection, that is, π is a *permutation* of the set $[M]$. It is sufficient to show that the image $\pi([M])$ is a set of distinct elements. First assume the contrary, i.e., suppose $\pi(l) = \pi(l') = m$ for two distinct elements $l, l' \in [M]$. Then

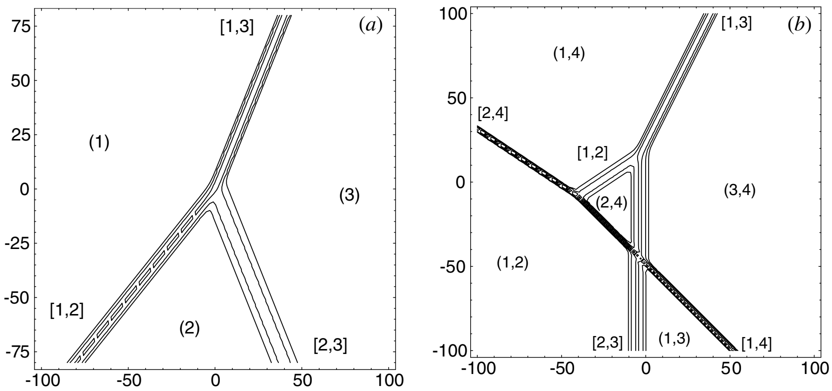

Figure 2. Dominant phase combinations in different regions of the (x, y)-plane (labeled by the indices in parentheses, i.e. $(i, j) = \theta(i, j)$) and the asymptotic line solitons (labeled by the indices in square brackets) for two different line-soliton solutions: (*a*) a (2, 1)-soliton with $(k_1, k_2, k_3) = (-1, 0, \frac{1}{2})$ at $t = 0$; (*b*) a (2, 2)-soliton with $(k_1, \ldots, k_4) = (-1, -\frac{1}{2}, \frac{1}{2}, 2)$ at $t = 0$.

consider the dominant phase combinations associated with the single-phase transitions (see proposition 2.4(i)) starting with the $\theta(e_1, \ldots, e_N)$ as $x \to -\infty$, proceeding clockwise, and finally back to $\theta(e_1, \ldots, e_N)$ after a complete revolution. There are altogether M such single-phase transitions: N transitions as $y \to \infty$ corresponding to the pivot indices $\{e_1, \ldots, e_N\}$, and $M - N$ transitions as $y \to -\infty$ corresponding to the non-pivot indices $\{g_1, \ldots, g_{M-N}\}$. Without loss of generality, if we assume that the transition $l \to m$ occurs before $l' \to m$, then there must be an intermediate $m \to m'$ transition for some $m' \in [M]$ in between those two single-phase transitions. Now if $\theta_m \in \theta(e_1, \ldots, e_N)$ then the transition $m \to m'$ must occur before the $l \to m$ transition can take place. Consequently, the intermediate $m \to m'$ transition can *not* occur as each transition occurs only once during a complete revolution in the (x, y)-plane. On the other hand, if $\theta_m \notin \theta(e_1, \ldots, e_N)$ then the $m \to m'$ transition must occur after the $l' \to m$ transition, and again the intermediate $m \to m'$ transition cannot take place. Either way, we reach a contradiction implying that $\pi : [M] \to [M]$ is one-to-one, and therefore a bijection, thus proving our claim. Note also that π has no fixed point because $\pi(e_n) > e_n, n = 1, \ldots, N$ and $\pi(g_n) < g_n, n = 1, \ldots, M - N$. A permutation π with no fixed point is called a *derangement*, and an element $l \in [M]$ is called an *excedance* of π if $\pi(l) > l$. We can summarize the above discussions as follows.

Proposition 2.9. *The pairing map π defined by equation (2.4) is a derangement of $[M]$ with N excedances which are given by the pivot indices $\{e_1, \ldots, e_N\}$ of the coefficient matrix A in RREF.*

In example 2.7, the pivot index $e_1 = 1$, and the non-pivot indices $g_1 = 2, g_2 = 3$ for the coefficient matrix A, with $M = 3$. The soliton pairings are: [1, 3] as $y \to \infty$, and [1, 2], [2, 3] as $y \to -\infty$. The corresponding pairing map from equation (2.4) is given by

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

in the bi-word notation of permutation, and π has only one excedance: $\{1\}$. In example 2.8, $M = 4$, the pivot and non-pivot indices are $e_1 = 1, e_2 = 2$ and $g_1 = 3, g_2 = 4$, respectively.

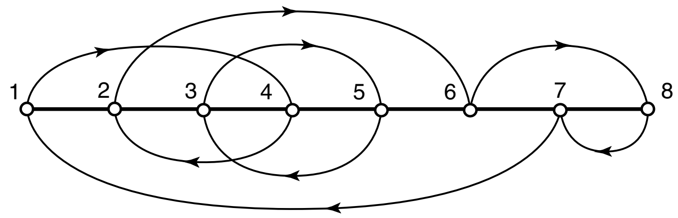

Figure 3. The open chord diagram for $\pi = (46\,523\,817)$.

The pairing map

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

corresponds to the line solitons [1, 3], [2, 4] as $y \to \infty$, and [2, 3], [1, 4] as $y \to -\infty$ with excedance set $\{1, 2\}$. The pairing map π can also be represented by an open chord diagram associated with permutations (see [8]). For example, shown in figure 3 is the diagram for $\pi = (46\,523\,817)$ (using one-line notation for permutations).

The four upper chords in figure 3 originate from the four excedances $\{1, 2, 3, 6\}$ with the right arrows indicating the increasing order of the numbers. The upper chords correspond to the asymptotic line solitons for $y \to \infty$ with the pairings [1, 4], [2, 6], [3, 5], [6, 8]. Similarly, the lower chords with reversed arrows indicate the line solitons for $y \to -\infty$, namely, [1, 7], [2, 4], [3, 5], [7, 8]. Thus, this chord diagram represents a (4, 4)-soliton solution of KPII. Note here that the [3, 5]-soliton appears both in $y \to \pm\infty$ which corresponds to a 2-cycle (35) in the permutation π.

Proposition 2.9 furnishes a combinatorial characterization of the (N_-, N_+)-soliton solutions of KPII as explained below.

Definition 2.10. *Let* $S_+ := \{[e_n, j_n]\}_{n=1}^{N}$ *and* $S_- := \{[i_n, g_n]\}_{n=1}^{M-N}$ *denote the index sets labeling the asymptotic line solitons as* $y \to \infty$ *and as* $y \to -\infty$*, respectively. Then two* (N_-, N_+)*-soliton solutions of KPII are defined to be in the same equivalence class if their asymptotic line solitons are labeled by the identical sets* $S_\pm$ *of index pairs, where* $|S_+| := N_+ = N$ *and* $|S_-| := N_- = M - N$*.*

Each equivalence class of (N_-, N_+)-soliton solutions of KPII is uniquely determined by a derangement π as defined in proposition 2.9. Therefore, each equivalence class of soliton solutions is also associated with a unique open chord diagram with $N_+ = N$ upper chords and $N_- = M - N$ lower chords. Furthermore, in view of remark 2.3, each derangement of $[M]$ gives a unique parametrization of a TNN Grassmann cell in $Gr^+(N, M)$, denoted by $W(\pi)$, and whose dimension can be computed from the number of crossings of the corresponding chord diagram [24, 28]. In particular, the dimension of the cell $W(\pi)$ associated with a TNN matrix satisfying condition 2.2 is given by [7]

$$\dim W(\pi) = N + C_+(\pi) + C_-(\pi),$$

where the number of crossings $C_\pm(\pi)$ in the chord diagram is defined by [8]

$$C_\pm(\pi) = \sum_{i=1}^{M} C_\pm(i), \qquad \text{where} \quad \begin{cases} C_+(i) := \{j : j < i < \pi(j) < \pi(i)\}, \\ C_-(i) := \{j : j > i \geqslant \pi(j) > \pi(i)\}. \end{cases} \tag{2.5}$$

(Note that the above definitions of $C_\pm(i)$ are switched from those given in [8].) For the example in figure 3, the chord diagram associated with $\pi = (46\,523\,817)$ has $C_+(\pi) = 2$, $C_-(\pi) = 2$, and the corresponding Grassmann cell in $Gr^+(4, 8)$ has dimension $8 = 4 + 2 + 2$. The top (maximal) dimensional cell in $Gr^+(4, 8)$ of dimension $16 = 4 \times 4$ corresponds to a chord diagram with the maximum number of crossings, i.e. $\pi_{\text{top}} = (56\,781\,234) = (15)(26)(37)(48)$ with $C_+(\pi_{\text{top}}) = C_-(\pi_{\text{top}}) = 6$, and $\dim W(\pi_{\text{top}}) = 16 = 4 + 6 + 6$. An equivalence class of (N_-, N_+)-soliton solutions of KPII with a given pairing map π can thus be associated with a unique TNN Grassmann cell $W(\pi)$, and the number of free variables parametrizing the (N_-, N_+)-soliton solution space is given by $\dim W(\pi)$. We next proceed to calculate the total number of (N_-, N_+)-soliton equivalence classes for given values of M and N, and enumerate these equivalence classes according to the dimensions of their solution spaces.

Let $\mathcal{S}_M$ denote the permutation group of $[M]$, and let $\mathcal{D}_M \subset \mathcal{S}_M$ be the set of all derangements of $[M]$. Then the derangements can be enumerated according to the number of excedances $e(\pi)$ of π by the generating polynomial

$$D_M(p) = \sum_{\pi\in\mathcal{D}_M} p^{e(\pi)} = \sum_{N=1}^{M-1} D_{N,M}\, p^N, \qquad M \geqslant 1,$$

where the coefficients $D_{N,M}$ denote the number of derangements of $[M]$ with N excedances. The total number of derangements of $\mathcal{S}_M$ is then given by $|\mathcal{D}_M| = D_M(1)$. The explicit formula for the derangement polynomial $D_M(p)$ is obtained from the exponential generating function [25] with $D_0(p) := 1$,

$$D(p, z) = \sum_{M=0}^{\infty} D_M(p)\frac{z^M}{M!} = \frac{1-p}{\mathrm{e}^{zp} - p\mathrm{e}^{z}}. \tag{2.6}$$

The first few polynomials are given by $D_1(p) = 0$, $D_2(p) = p$, $D_3(p) = p + p^2$, $D_4(p) = p + 7p^2 + p^3$. Moreover, the polynomials $D_M(p)$ are *symmetric*, that is, its coefficients satisfy

$$D_{N,M} = D_{M-N,M}, \qquad N = 1, 2, \ldots, M-1. \tag{2.7}$$

This and various other properties of the derangement polynomial $D_M(p)$ can be found in [25]. Note that equation (2.7) is equivalent to the relation

$$p^M D_M(p^{-1}) = D_M(p),$$

which in turn follows from the symmetry

$$D(p^{-1}, zp) = D(p, z)$$

of the exponential generating function, and can be verified directly from equation (2.6).

The above formulae give the number $D_{N,M}$ of equivalence classes for the (N_-, N_+)-solitons of KPII for a given $M = N_- + N_+$ and $N = N_+$. Thus, when $M = 3$, there are only two classes of line-soliton solutions, namely, the (2, 1)-soliton as in example 2.7, and also (1, 2)-solitons which are related to the (2, 1)-solitons by the inversion symmetry: $(x, y, t) \to (-x, -y, -t)$. When $M = 4$, there are one type each of the (3, 1)- and (1, 3)-soliton solutions related via the inversion symmetry, but also seven distinct types of (2, 2)-soliton solutions. It is also possible to obtain a further refinement of the total number of soliton equivalence classes by introducing a q-analog of the derangement number $D_{N,M}$, namely,

$$D_{N,M}(q) = \sum_{r=N}^{N(M-N)} D_{r,N,M}\, q^r,$$

where $D_{r,N,M}$ is the number of derangements of $[M]$ with N excedances, and $r-N$ crossings as defined in equation (2.5). Then, $D_{N,M}(q)$ is the generating polynomial for the Grassmann cells in $Gr^+(N,M)$ corresponding to the irreducible TNN matrices satisfying condition 2.2, and $D_{r,N,M}$ is the number of those TNN cells of dimension r. The upper limit $N(M-N)$ in the sum gives the dimension of the top cell in $Gr^+(N,M)$. Equivalently, $D_{r,N,M}$ then gives the number of (N_-,N_+)-soliton equivalence classes with $N_+=N$, $N_-=M-N$, and with r free parameters. Moreover, the total number of (N_-,N_+)-soliton equivalence classes is given by $D_{N,M}(q=1)=D_{N,M}$.

It is interesting to note that $D_{N,M}(q)$ is related to a q-analog of the Eulerian number [28],

$$E_{N,M}(q)=q^{M-N^2-N}\sum_{i=0}^{N-1}(-1)^i[N-i]_q^M q^{Ni}\left(\binom{M}{i}q^{N-i}+\binom{M}{i-1}\right),$$

where $[N]_q:=1+q+q^2+\cdots+q^{N-1}$ is the q-analog of the number N. The polynomial $E_{N,M}(q)$ was recently introduced in [28] where a rank generating function for the cells in $Gr^+(N,M)$ was derived by building on the work of [24]. It follows from [8, 28] that the coefficient of q^r in $E_{N,M}(q)$ is the number of permutations of $[M]$ with N *weak* excedances, and whose chord diagrams have $r-N$ crossings as defined in equation (2.5). Note that $l\in[M]$ is called a *weak* excedance of a permutation $\pi\in\mathcal{S}_M$ if $\pi(l)\geqslant l$. The following result gives the relation between the polynomials $D_{N,M}(q)$ and $E_{N,M}(q)$.

Proposition 2.11. *For fixed $N_+=N, N_-=M-N$, the generating polynomial for the (N_-,N_+)-soliton equivalence classes according to the dimension of their solution spaces is given by*

$$D_{N,M}(q)=\sum_{j=0}^{N-1}(-1)^j\binom{M}{j}E_{N-j,M-j}(q),$$

where $E_{k,n}(q)$ are the Eulerian polynomials defined above.

Proof. Let $E(N,M)$ denote the set of all permutations of $[M]$ with N weak excedances, and let $D(N,M)\subset\mathcal{D}_M$ denote the derangements of $[M]$ with N excedances. Then the polynomials $E_{N,M}(q)$ and $D_{N,M}(q)$ are given in terms of the number of crossings $c(\pi)$ of the permutation π as

$$E_{N,M}(q)=q^N\sum_{\pi\in E(N,M)}q^{c(\pi)},\qquad D_{N,M}(q)=q^N\sum_{\pi\in D(N,M)}q^{c(\pi)}.$$

Note that for each $n\leqslant N-1$, an element of $E(N,M)$ can be obtained by adding n fixed points to the corresponding element of the derangement set $D(N-n,M-n)$. Then for $N'=N-n, M'=M-n$,

$$E_{N,M}(q)=q^N\sum_{\substack{S\subset[M],\\|S|=n}}\sum_{\pi\in D(N',M')}q^{c(\pi)}=\sum_{n=0}^{N-1}\binom{M}{n}D_{N-n,M-n}(q).$$

Inverting the above formula yields the desired result. □

Proposition 2.11 provides an explicit formula for enumerating the (N_+,N_-)-soliton equivalence classes according to the dimensions of the associated Grassmann cells in $Gr^+(N,M)$. For example, when $M=4$ and $N=2$, proposition 2.11 yields $D_{2,4}(q)=q^4+4q^3+2q^2$. This implies that there are one cell of dimension 4 (top cell), four cells of dimension 3 and two cells of dimension 2. The $Gr^+(2,4)$ case will be discussed in the following section.

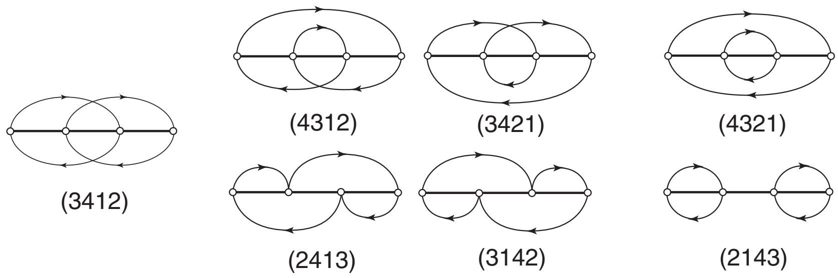

Figure 4. The open chord diagrams for the seven equivalence classes of (2, 2)-soliton solutions.

It turns out that the line-soliton solutions of KPII possess several other combinatorial properties which play significant roles in their classification scheme. Some of these properties were addressed in [15] (see also [3]). In this paper, we will present these combinatorial structures underlying the line-soliton solutions from an algebraic perspective in section 4.

3. (2, 2)-Soliton solutions

In this section, we study those line-soliton solutions of KPII which admit a pair of asymptotic line solitons as $|y| \to \infty$. But in general, the pair of line solitons as $y \to \infty$ differ from those as $y \to -\infty$ in their amplitudes and directions. We call these the (2, 2)-soliton solutions, which include the two-soliton solutions as well. According to the results of section 2.2 regarding the asymptotic properties of the τ-function, these solutions are specified by prescribing the distinct phase parameters $\{k_1, \dots, k_4\}$ and the 2×4 coefficient matrix A. The τ-function of any (2, 2)-soliton is given by

$$\tau(x, y, t) = \sum_{1 \leqslant r < s \leqslant 4} (k_s - k_r) A(r, s)\, e^{\theta_r + \theta_s}, \tag{3.1}$$

where $A(r, s)$ denotes the 2×2 non-negative minors of the matrix A.

3.1. Classification of (2, 2)-soliton solutions

For a given set of phase parameters $\{k_1, \dots, k_4\}$, all possible equivalence classes of (2, 2)-soliton solutions can be completely enumerated by the derangements of the index set [4] with two excedances. Recall from section 2.3 that there are altogether seven distinct equivalence classes of (2, 2)-soliton solutions given by the coefficient $D_{2,4}$ of p^2, in the derangement polynomial $D_4(p)$. The seven equivalence classes can be further enumerated by the q-derangement number $D_{2,4}(q) = q^4 + 4q^3 + 2q^2$ according to the number of free parameters spanning the solution space of each equivalence class. The coefficients 1, 4, 2 in $D_{2,4}(q)$ correspond to the number of (2, 2)-soliton equivalence classes spanned, respectively, by 4, 3 and 2 free parameters. This is illustrated in figure 4 by the open chord diagrams associated with the (2, 2)-soliton solutions.The number sequence below each diagram in figure 4 is the one-line notation for the corresponding permutation π which also designates a certain TNN cell $W(\pi)$ in $Gr^+(2, 4)$. The left diagram with two crossings corresponds to the top cell of dimension 4; the middle four diagrams with one crossing in each correspond to the dimension 3 cells; while the last two diagrams have no crossing, and correspond to the cells of dimension 2 in $Gr^+(2, 4)$. We verify these combinatorial results below by directly analyzing the (2, 2)-soliton coefficient matrices A which represent the TNN cells in $Gr^+(2, 4)$.

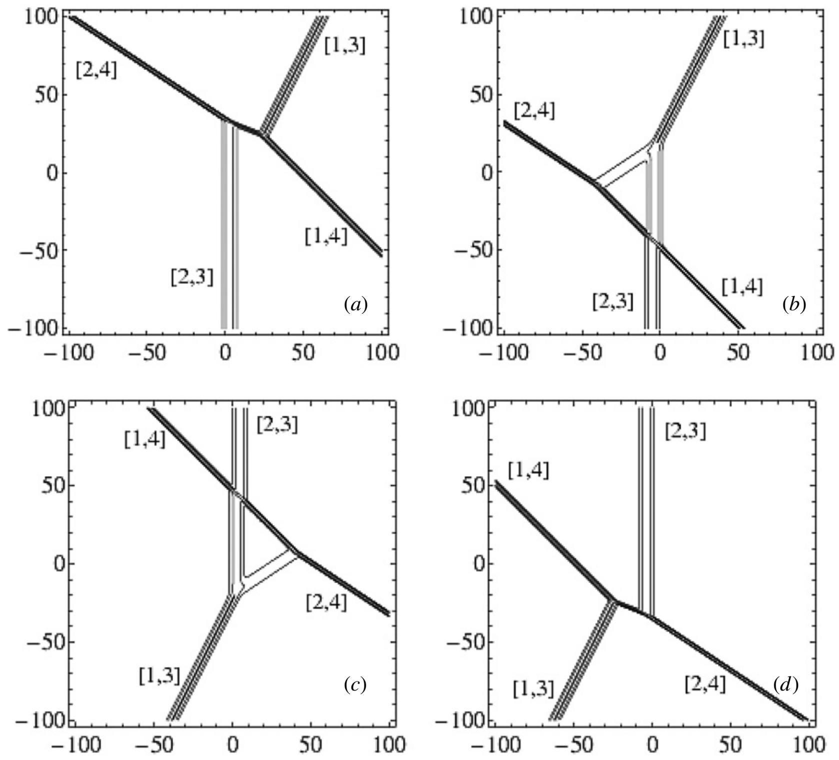

Figure 5. Two different (2, 2)-soliton solutions of KPII with phase parameters $(k_1, \ldots, k_4) = \left(-1, -\frac{1}{2}, \frac{1}{2}, 2\right)$. The first solution with $\pi = (3421)$ at $t = -16$ and $t = 16$ is shown, respectively, in (*a*) and (*b*); while (*c*) and (*d*) represent the second solution with $\pi = (4312)$ at $t = -16$ and $t = 16$. Note the spacetime reversal symmetry relating the two solutions in (*a*), (*d*) and in (*b*), (*c*).

The (2, 2)-soliton solutions arise from one of the following two irreducible coefficient matrices in RREF:

$$A_1 = \begin{pmatrix} 1 & 0 & -a & -b \\ 0 & 1 & c & d \end{pmatrix} \quad \text{or} \quad A_2 = \begin{pmatrix} 1 & a & 0 & -b \\ 0 & 0 & 1 & d \end{pmatrix} \tag{3.2}$$

with arbitrary *non-negative* parameters a, b, c, d. We classify them according to the number of independent positive parameters in the coefficient matrix A_1 or in A_2. Note that the matrices in equations (3.2) identify certain Schubert cells of $Gr(2, 4)$. A further refinement of those Schubert cells is given by the TNN Grassmann cells classified below.

Four positive parameters. There is one such case, which corresponds to the matrix A_1 in equation (3.2) with $A_1(34) = bc - ad > 0$. Note that all six exponential terms are present in the τ-function in equation (3.1). From proposition 2.6, one finds that the pair of asymptotic line solitons are the *same* as $|y| \to \infty$ with index pairs ([1, 3], [2, 4]). The associated pairing map given by the permutation is $\pi = (3412) = (13)(24)$. Thus, this case corresponds to an equivalence class of two-soliton solutions where each asymptotic line soliton as $y \to \infty$ has identical amplitude and direction to another asymptotic line soliton as $y \to -\infty$. The two-soliton solutions will be discussed further in the following subsection.

Three positive parameters. There are two possibilities, namely, A_2 with nonzero parameters a, b, d, and A_1 with three free parameters. Consider A_1 first. In this case neither b nor c

can vanish in A_1, else $A_1(34) < 0$. Hence, a three-parameter family of the matrix A_1 with non-negative maximal minors arises in three possible ways, namely, $a = 0$, or $d = 0$, or $A_1(34) = 0$ but neither of a, b, c, d is zero. That is,

$$(i)\ A_1 = \begin{pmatrix} 1 & 0 & 0 & -b \\ 0 & 1 & c & d \end{pmatrix}, \qquad (ii)\ A_1 = \begin{pmatrix} 1 & 0 & -a & -b \\ 0 & 1 & c & 0 \end{pmatrix},$$
$$(iii)\ A_1 = \begin{pmatrix} 1 & 0 & -a & -b \\ 0 & 1 & c & d \end{pmatrix} \qquad \text{with} \quad A_1(34) = 0. \tag{3.3}$$

Case (i) in equation (3.3) gives rise to a (2, 2)-soliton solution with asymptotic line-soliton pairs ([2, 4], [1, 3]) as $y \to \infty$, and ([2, 3], [1, 4]) as $y \to -\infty$. The associated pairing map is given by $\pi_1 = (3421)$. The second matrix in equation (3.3) corresponds to another (2, 2)-soliton solution whose asymptotic line-soliton pairs are ([1, 4], [2, 3]) as $y \to \infty$, and ([1, 3], [2, 4]) as $y \to -\infty$. These two solutions are related via spacetime reversal $(x, y, t) \to (-x, -y, -t)$ as shown in figure 5. Furthermore, the pairing map for the second solution $\pi_2 = (4312)$ satisfies $\pi_2 = \pi_1^{-1}$. The last case in equation (3.3) yields a (2, 2)-soliton with asymptotic line soliton pairs ([2, 4], [1, 2]) as $y \to \infty$, and ([1, 3], [3, 4]) as $y \to -\infty$, which determine the pairing map $\pi_3 = (2413)$. Like the previous two cases, case (iii) is also related via a spacetime reversal to another (2, 2)-soliton solution which is associated with the coefficient matrix A_2 with three positive parameters as in equation (3.2). Thus, A_2 corresponds to the asymptotic line-soliton pairs ([1, 3], [3, 4]) as $y \to \infty$ and ([2, 4], [1, 2]) as $y \to -\infty$, with the pairing map $\pi_4 = (3142) = \pi_3^{-1}$.

Two positive parameters. It should be clear from equation (3.2) that if either $b = 0$ or $c = 0$ in A_1, it cannot satisfy condition 2.2. So, A_1 will contain precisely two free parameters if and only if $a = d = 0$. Similarly, A_2 in equation (3.2) will have two free parameters if and only if $b = 0$. Each of the nonzero parameters b, c in A_1, and a, d in A_2 can be rescaled to unity without loss of generality, by utilizing the gauge freedom described in property 2.1(v). The matrices resulting in this way from A_1 and A_2 are denoted by A_{P} and A_{O}, respectively, and are presented in equation (3.4) below. Each coefficient matrix produces an equivalence class of two-soliton solutions as $|y| \to \infty$ with asymptotic soliton pairs ([1, 4], [2, 3]) for A_{P} with $\pi_{\mathrm{P}} = (4321)$, and ([1, 2], [3, 4]) for A_{O} with $\pi_{\mathrm{O}} = (2143)$.

3.2. Equivalence classes of two-soliton solutions

As noted in the previous subsection, there are three types of two-soliton solutions. They will be referred to as O-, T- and P-types (following the terminology introduced in [15]) below. These are identified by the canonical coefficient matrices

$$A_{\mathrm{O}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad A_{\mathrm{T}} = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & x_1 & x_2 \end{pmatrix}, \qquad A_{\mathrm{P}} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \tag{3.4}$$

with $x_1 > x_2 > 0$ in A_{T}. Up to rescaling of columns, the above matrices were also obtained as special subcases of the classification presented in section 3.1. A_{O} describes O-type two-solitons, with asymptotic line solitons [1, 2] and [3, 4], A_{T} describes T-type resonant two-solitons with asymptotic line solitons [1, 3] and [2, 4] and A_{P} describes P-type two-solitons with asymptotic line solitons [1, 4] and [2, 3]. A distinctive feature of the pairing map π of a two-soliton solution is the fact that π is an involution satisfying $\pi = \pi^{-1}$. As a result, π can be expressed as a product of disjoint 2-cycles (see, e.g. [5]). For the permutation group $\mathcal{S}_4$, there are only three such involutions corresponding to the total number of disjoint partitions of [4] into two pairs. In cycle notation, these involutions are given by $\pi_{\mathrm{O}} = (12)(34)$, $\pi_{\mathrm{T}} = (13)(24)$

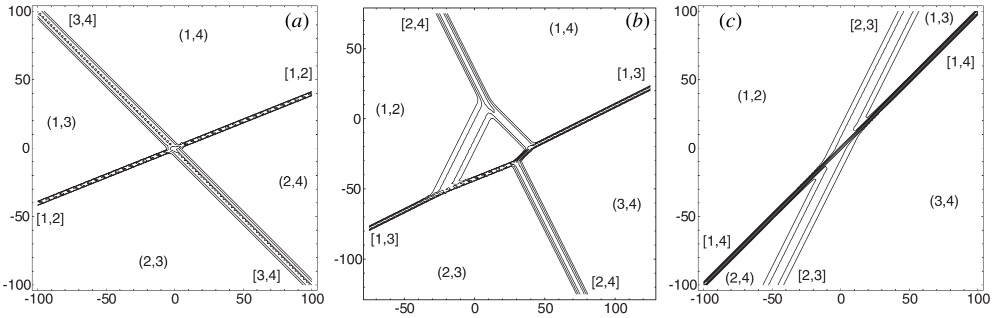

Figure 6. Three different two-soliton solutions of KPII with the same phase parameters $(k_1, \ldots, k_4) = (-2, -\frac{1}{2}, 0, 1)$, illustrating the three equivalence classes: (*a*) O-type two-soliton solution, yielding $(c_1, c_2) = (-\frac{5}{2}, 1)$ and $(a_1, a_2) = (\frac{3}{2}, 1)$; (*b*) T-type two-soliton solution, yielding $(c_1, c_2) = (-2, \frac{1}{2})$ and $(a_1, a_2) = (2, \frac{3}{2})$; (*c*) P-type two-soliton solution, yielding $(c_1, c_2) = (-1, -\frac{1}{2})$ and $(a_1, a_2) = (3, \frac{1}{2})$. (Note that for this P-type solution $c_{[1,4]} < c_{[2,3]}$.)

and $\pi_P = (14)(23)$ for the O-, T- and P-type two-soliton solutions, respectively. The 2-cycles are also evident from the chord diagrams illustrated in figure 4 where the extreme left (3412) diagram corresponds to the T-type two-soliton, while the extreme right diagrams (4321) and (2143), respectively, represent the P- and O-type two-soliton equivalence classes.

Figure 6 shows a representative solution for each of the three equivalence classes with the same phase parameters $k_1, \ldots, k_4$. Note that the O- and P-type solitons interact via an X-junction (ignoring the phase shifts), while the T-type solitons interact via four Y-junctions connecting the four asymptotic line solitons to four intermediate segments. Each of these intermediate segments satisfy the nonlinear dispersion relation (1.7), and at each Y-junction the resonance condition (1.9) is satisfied. Thus each intermediate segment is also a line-soliton. For example, in figure 6(*b*) the asymptotic line soliton $[1, 3]$ (as $y \to -\infty$) forms the intermediate line solitons $[1, 2]$ and $[2, 3]$ at the bottom left Y-junction. The line-soliton $[2, 3]$ connects with the asymptotic line-soliton $[2, 4]$ (as $y \to \infty$) and the line-soliton $[1, 2]$ connects with the asymptotic line soliton $[2, 4]$ (as $y \to -\infty$). Similarly, the asymptotic line soliton $[1, 3]$ (as $y \to \infty$) forms the intermediate line solitons $[1, 4]$ and $[3, 4]$ at the top right Y-junction. The line-soliton $[3, 4]$ connects with the asymptotic line-soliton $[2, 4]$ (as $y \to \infty$) and the line-soliton $[1, 4]$ connects with the asymptotic line soliton $[2, 4]$ (as $y \to -\infty$).

An important distinction among the three types of two-soliton solutions is that they cover different regions of the soliton parameter space. Suppose (a_1, c_1) and (a_2, c_2) are the soliton parameters of the asymptotic line solitons of each type with the same set of distinct phase parameters. Since the phase parameters are ordered: $k_1 < \cdots < k_4$, the soliton parameters satisfy the following relations which can be easily verified using equations (1.10).

(i) For an O-type two-soliton solution $c_2 > c_1$ and $c_2 - c_1 > a_1 + a_2$.
(ii) For a T-type two-soliton solution $c_2 > c_1$ and $|a_1 - a_2| < c_2 - c_1 < a_1 + a_2$.
(iii) For a P-type two-soliton solution $a_2 > a_1$ and $|c_2 - c_1| < a_2 - a_1$.
(iv) $(c_2 - c_1)_O > (c_2 - c_1)_T > |c_2 - c_1|_P$, $(a_1 + a_2)_O < (a_1 + a_2)_T = (a_1 + a_2)_P$ and $|a_2 - a_1|_O = |a_2 - a_1|_T < (a_2 - a_1)_P$.

Note that for O- and T-type solutions the soliton directions are ordered, while for P-type solutions the amplitudes are ordered. *Any* choice of the soliton parameters $\{(a_i, c_i) | a_i > 0\}_{i=1}^2$ would lead to one of the three types of two-soliton solutions provided that $\{c_1 \pm a_1, c_2 \pm a_2\}$

are distinct real numbers. Thus, the three types of two-soliton solutions divide the soliton parameter space into disjoint sectors bounded by the hyperplanes $|c_2 - c_1| = a_1 + a_2$ and $|c_2 - c_1| = |a_1 - a_2|$. At each boundary between two disjoint regions of the soliton parameter space, two of the phase parameters coincide. In such a situation, it can be shown (by taking suitable limits) that the two-soliton solution degenerates into a Y-junction [4, 15].

Yet another difference is in the phase shifts experienced by the asymptotic line solitons of each type. The position of an asymptotic line-soliton $[i, j]$ is determined by the dominant phase combinations across the soliton and from the asymptotic formulae (2.3*a*) and (2.3*b*). The pairs of dominant phase combinations across the soliton $[i, j]$ as $y \to \infty$ and as $y \to -\infty$ are *distinct* from each other (see, e.g. figure 6). This fact gives rise to the phase (position) shift $\Delta_{ij} = \delta^+_{ij} - \delta^-_{ij}$ for the asymptotic line-soliton $[i, j]$. Since the asymptotic positions $\delta^{\pm}_{ij}$ have the *same* linear dependence on time t, the phase shift Δ_{ij} is independent of t. The dominant phase combinations across an asymptotic line-soliton in each of the three types of two-soliton solutions can be determined from the asymptotic analysis of section 2 and are shown in figure 6. Then the soliton phase shifts computed using the parameters (a_1, c_1), (a_2, c_2) and the canonical coefficient matrices in equation (3.4) are given by

$$\begin{aligned} \Delta_{\rm O} &= \log\frac{(c_1-c_2)^2-(a_1-a_2)^2}{(c_1-c_2)^2-(a_1+a_2)^2} = \Delta_{\rm P}, \\ \Delta_{\rm T} &= \log\frac{(c_1-c_2)^2-(a_1-a_2)^2}{(c_1-c_2)^2-(a_1+a_2)^2} + \log\left(\frac{x_1}{x_2}-1\right). \end{aligned} \tag{3.5}$$

The phase shifts experienced by the pair of asymptotic line solitons are of opposite signs in each of the three cases. Above, we assumed that $c_1 \neq c_2$ (i.e., the solitons are not parallel), then Δ denotes the phase shift of the soliton with the direction parameter c_2 where $c_2 > c_1$. For non-resonant (O- and P-types) two-soliton solutions, phase shift expressions in (3.5) have only one and the same term which depends symmetrically on the soliton parameters. However, it is easy to verify from the various inequalities mentioned above, among the soliton parameters that $\Delta_{\rm O}$ is always *positive* while $\Delta_{\rm P}$ is always *negative*. For resonant two-soliton solutions there is an additional term that depends on the free parameters x_1, x_2 of the coefficient matrix $A_{\rm T}$ in equation (3.4). In this case, the phase shift $\Delta_{\rm T}$ is not sign-definite unlike the other two cases.

4. Duality and *N*-soliton solutions

In the previous section, we described two types of equivalence classes of line-soliton solutions of KPII: (i) the (2, 2)-solitons where the sets $S_{\pm}$ (cf definition 2.10) of asymptotic line solitons as $y \to \pm\infty$ are distinct and (ii) the two-soliton solutions with $S_- = S_+$. Furthermore, we noted that there are pairs of distinct equivalence classes of (2, 2)-soliton solutions related via spacetime reversal (cf figure 5). The general (N_-, N_+)-soliton solutions can be also categorized in a similar fashion according to whether the sets of asymptotic line solitons S_+ and S_- are distinct, or if $S_- = S_+$. In the first case, pairs of equivalence classes of solutions are related by spacetime reversal, while the latter corresponds to the special case of *N*-soliton solutions to be discussed in this section.

4.1. Duality of line solitons

The KPII equation (1.1) is invariant under the inversion $(x, y, t) \to (-x, -y, -t)$. Consequently, if $u(x, y, t)$ is a (N_-, N_+)-soliton solution of KPII with given sets $S_{\pm}$ of asymptotic line solitons and with $N_- = M - N$ and $N_+ = N$, then $u(-x, -y, -t)$ is a

$(N, M-N)$-soliton solution with *reversed* sets $S_\mp$ of asymptotic line solitons. We refer to the solutions $u(x, y, t)$ and $u(-x, -y, -t)$, as well as their respective equivalence classes as *dual* to each other. Let $\tau_{N,M}(x, y, t)$ denote the τ-function in equation (2.2), generating the solution $u(x, y, t)$ via equation (1.2), then the solution $u(-x, -y, -t)$ will be generated by $\tau_{N,M}(-x, -y, -t)$ as equation (1.2) remains invariant under $(x, y, t) \to (-x, -y, -t)$. Note, however, that $\tau_{N,M}(-x, -y, -t)$ does not have the form given by equation (2.2), but it is indeed possible to construct a certain τ-function $\tau_{M-N,M}(x, y, t)$ from $\tau_{N,M}(-x, -y, -t)$ that is dual to $\tau_{N,M}(x, y, t)$. We describe below how to construct the function $\tau_{M-N,M}$ from $\tau_{N,M}$.

First, we obtain a coefficient matrix for the τ-function $\tau_{M-N,M}$ from the $N \times M$ matrix A associated with $\tau_{N,M}(x, y, t)$. Since A is of full rank, its rows form a basis for a N-dimensional subspace W of $\mathbb{R}^M$. Let $W^\perp$ be the orthogonal complement of W with respect to the standard inner product on $\mathbb{R}^M$, and let A' be a $(M-N) \times M$ matrix whose rows form a basis for $W^\perp$. Clearly, A' is not unique, but a particular choice for A' is as follows: suppose the pivot and non-pivot columns of A in RREF are represented by the identity matrix $\mathcal{I}_N$ and the $N \times (M-N)$ matrix G, respectively, then

$$A = [\mathcal{I}_N, G]P \quad \Rightarrow \quad A' = [-G^T, \mathcal{I}_{M-N}]P, \tag{4.1}$$

where G^T is the matrix transpose of G, and P is a $M \times M$ permutation matrix satisfying $P^T = P^{-1}$. It can be directly verified from equation (4.1) that $AA'^T = 0$, which constitute the orthogonality relations among the row vectors of A and A'. Moreover, it was shown in [2] that the $N \times N$ minors of A and the $(M-N) \times (M-N)$ minors of the matrix

$$B' := A'E, \qquad E := \mathrm{diag}(-1, 1, -1, \ldots, \pm 1)$$

are related as

$$A(m_1, \ldots, m_N) = (-1)^\sigma \det(P) B'(l_1, \ldots, l_{M-N}),$$

where $\sigma = M(M+1)/2 + N(N+1)/2$ and $\{m_1, m_2, \ldots, m_N\}$ is the complement of $\{l_1, l_2, \ldots, l_{M-N}\}$ in $[M]$. The factor $(-1)^\sigma \det(P) = \pm 1$, which depends only on A, can be rescaled by an orthogonal transformation $B' \to OB'$, $\det(O) = \pm 1$, so that the maximal minors of the rescaled matrix

$$B := OB' = O[-G^T, \mathcal{I}_{M-N}]PE \tag{4.2a}$$

satisfy the precise complementarity conditions

$$A(m_1, \ldots, m_N) = B(l_1, \ldots, l_{M-N}). \tag{4.2b}$$

Hence, if all maximal minors of A are non-negative, then the same holds for B.

The $(M-N) \times M$ matrix B plays the role of a coefficient matrix for the τ-function $\tau_{M-N,M}(x, y, t)$ which is related to the function $\tau_{N,M}(-x, -y, -t)$. Indeed, if we initially set $\theta_{m,0} = 0, \forall m = 1, \ldots, M$ in equation (2.2) using property 2.1(v), then under the transformation $(x, y, t) \to (-x, -y, -t)$, equation (2.2) yields

$$\tau_{N,M}(-x, -y, -t) = \exp[-\theta(1, \ldots, M)]\tau'(x, y, t).$$

Using equation (4.2b) and taking the sum over the complementary indices $l_1, \ldots, l_{M-N}$ (instead of $m_1, \ldots, m_N$), the function $\tau'(x, y, t)$ can be expressed as

$$\tau'(x, y, t) = \sum_{1 \leqslant l_1 < \ldots < l_{M-N} \leqslant M} V(m_1, \ldots, m_N) B(l_1, \ldots, l_{M-N}) \exp[\theta(l_1, \ldots, l_{M-N})], \tag{4.3}$$

where

$$V(m_1, \ldots, m_N) = \prod_{1 \leqslant s < r \leqslant N} \left(k_{m_r} - k_{m_s}\right)$$

are the Van der Monde coefficients as in equation (2.2). Next, if we introduce new phase constants by

$$\theta_{m,0} = \sum_{r\neq m} \ln|k_r - k_m|, \qquad m = 1, \ldots, M,$$

which satisfy the identity

$$\exp\left[\theta_{l_1,0} + \cdots + \theta_{l_{M-N},0}\right] = \frac{V(l_1, \ldots, l_{M-N})V(1, 2, \ldots, M)}{V(m_1, \ldots, m_N)},$$

and make the replacement: $\theta_m \to \theta_m + \theta_{m,0}$ in equation (4.3), we finally obtain the dual τ-function

$$\tau_{M-N,M}(x, y, t) := \frac{\tau'(x, y, t)}{V(1, \ldots, M)} = \sum_{1\leqslant l_1<\ldots<l_{M-N}\leqslant M} V(l_1, \ldots, l_{M-N})B(l_1, \ldots, l_{M-N}) \times \exp[\theta(l_1, \ldots, l_{M-N})]. \qquad (4.4)$$

It is clear from equations (1.2) and (4.4) that the functions $\tau_{N,M}(-x, -y, -t)$, $\tau'(x, y, t)$ and $\tau_{M-N,M}(x, y, t)$ give rise to the same solution $u(-x, -y, -t)$ of KPII. Thus we have the following.

Proposition 4.1.

(i) The equivalence classes of solutions generated by the $N \times M$ coefficient matrix A and the $(M - N) \times M$ matrix B defined by equation (4.2a) are dual to each other. If a $(M - N, N)$-soliton solution $u(x, y, t)$ of KPII belongs to a certain equivalence class, then its dual equivalence class contains the $(N, M - N)$-soliton solution $u(-x, -y, -t)$.

(ii) Let $\{m_1, \ldots, m_N\}$ and $\{l_1, \ldots, l_{M-N}\}$ be a disjoint partition of the integer set $[M]$, then $\theta(m_1, \ldots, m_N)$ is a phase combination present in the τ-function $\tau_{N,M}$ if and only if $\theta(l_1, \ldots, l_{M-N})$ is a phase combination in the dual τ-function $\tau_{M-N,M}$.

(iii) If $\pi \in \mathcal{S}_M$ is the pairing map for a given equivalence class, then the pairing map for the dual equivalence class is given by π^{-1}.

Proposition 4.1 establishes a one-to-one correspondence between an equivalence class and its dual. Indeed from equation (2.7), we have the following result for the dual equivalent classes.

Proposition 4.2. *For given positive integers M, N, with $N < M$, the number of distinct equivalence classes of the $(M - N, N)$-soliton solutions are exactly the same as the number of dual equivalence classes of $(N, M - N)$-soliton solutions.*

Remark 4.3. The open chord diagrams of the pairing maps π and π^{-1} which correspond to a line-soliton equivalence class and its dual are related to each other via a reflection about the horizontal line together with reversing the direction of the chords. This is due to the fact that the excedance set of π^{-1} is given by $\{\pi(g_n)\}_{n=1}^{M-N}$, while the anti-excedance set is given by $\{\pi(e_n)\}_{n=1}^{N}$. This transformation on the chord diagrams can be regarded as the combinatorial analog of the inversion symmetry $(x, y, t) \to (-x, -y, -t)$ acting on the solutions $u(x, y, t)$ of the KPII line solitons.

When $M = 2N$, it follows from proposition 2.6 that $N_- = N_+ = N$, which leads to the (N, N)-soliton solutions. In this case the number of asymptotic line solitons as $y \to \infty$ and $y \to -\infty$ are the same, but in general, the amplitudes and directions of the line solitons will be different as seen for the (2, 2)-soliton examples in figure 5 of section 3. A particularly interesting subclass of the (N, N)-solitons are the N-soliton solutions which were introduced in section 1, and which are characterized by identical sets of asymptotic line solitons as $|y| \to \infty$, i.e., $S_- = S_+$. We discuss them next.

4.2. The N-soliton solutions

This special family of line-soliton solutions of KPII consists of equivalence classes that are invariant under the inversion symmetry $(x, y, t) \to (-x, -y, -t)$, that is, each equivalence class is its own dual so that both $u(x, y, t)$ and $u(-x, -y, -t)$ belong to the same equivalence class of solutions. This implies that for each N-soliton solution $u(x, y, t)$ the asymptotic line solitons arise in pairs, where each pair consists of a line soliton as $y \to \infty$ and another soliton as $y \to -\infty$, moreover, both line solitons have identical amplitude and direction. Thus, a (N_-, N_+)-soliton solution of KPII is a N-soliton solution if and only if it is *self-dual*, i.e., if and only if the index sets labeling the asymptotic line solitons satisfy $S_- = S_+$. The main features of the N-soliton solutions follow from propositions 2.6, 2.9 and 4.1 and definition 2.10. These are listed below.

Property 4.4.

(i) The τ-function of an N-soliton solution is expressed in terms of $2N$ distinct phase parameters and an $N \times 2N$ coefficient matrix A which satisfies condition 2.2. Then it follows from equation (4.2*b*) that the $N \times N$ minors of A satisfy the duality conditions

$$A(m_1, \ldots, m_N) = 0 \quad \Longleftrightarrow \quad A(l_1, \ldots, l_N) = 0, \tag{4.5}$$

where the indices $\{m_1, \ldots, m_N\}$ and $\{l_1, \ldots, l_N\}$ form a disjoint partition of integers $\{1, 2, \ldots, 2N\}$. That is, the phase combination $\theta(m_1, \ldots, m_N)$ is present in τ_N if and only if $\theta(l_1, \ldots, l_N)$ is present.

(ii) Each N-soliton solution has exactly N asymptotic line solitons as $y \to \pm\infty$. The line solitons as $y \to \infty$ and the line solitons as $y \to -\infty$ are identified by the *same* set of index pairs $\{[e_n, g_n]\}_{n-1}^N$ such that $\pi(e_n) = g_n$ and $\pi(g_n) = e_n$, with $e_n < g_n, n = 1, \ldots, N$. The sets $\{e_1, \ldots, e_N\}$ and $\{g_1, \ldots, g_N\}$ label, respectively, the pivot and non-pivot columns of the coefficient matrix A. Hence, they form a disjoint partition of the integer set $[2N]$.

(iii) The amplitude and direction of the nth asymptotic line soliton $[e_n, g_n]$ are the same as $y \to \pm\infty$, given by $a_n = k_{g_n} - k_{e_n}$ and $c_n = k_{g_n} + k_{e_n}$.

(iv) The pairing maps associated with N-soliton solutions are involutions of $\mathcal{S}_{2N}$ with no fixed points, defined by the set $\mathcal{I}_{2N} = \{\pi \in \mathcal{S}_{2N} | \pi^{-1} = \pi, \pi(i) \neq i, \forall i \in [2N]\}$. Such permutations can be expressed as products of N disjoint 2-cycles, and their chord diagrams are self-dual, i.e. symmetric about the horizontal axis (see, e.g., the two-soliton chord diagrams in section 3). The total number of such involutions is given by $|\mathcal{I}_{2N}| = (2N-1)!! = 1 \cdot 3 \cdot \cdots \cdot (2N-1)$ [5]. Hence, there are $(2N-1)!!$ distinct equivalence classes of N-soliton solutions.

Examples of the N-soliton solutions with special choices of the functions $\{f_n\}_{n=1}^N$ in equation (1.8) and the coefficient matrix A are given below.

Example 4.5. O-type N-soliton solutions. These are the well-known [10, 16] multi-soliton solutions of KPII constructed by choosing $\{f_n\}_{n=1}^N$ according to

$$f_n(x, y, t) = \mathrm{e}^{\theta_{2n-1}} + \mathrm{e}^{\theta_{2n}}, \qquad n = 1, \ldots, N.$$

The corresponding coefficient matrix is given by

$$A_{\mathrm{O}} = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix},$$

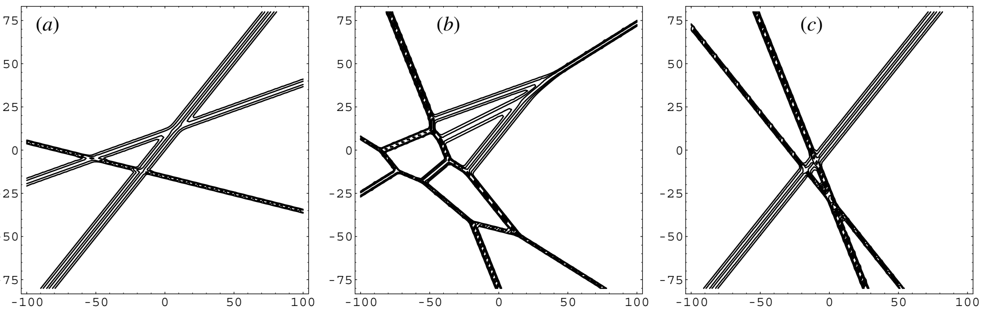

Figure 7. Three different three-soliton solutions of KPII with the same phase parameters $(k_1, \ldots, k_6) = (-3, -2, 0, 1, -\frac{3}{2}, 2)$, illustrating the three equivalence classes: (*a*) O-type, (*b*) T-type and (*c*) P-type three-soliton solutions.

with N pairs of identical columns at positions $\{2n-1, 2n\}, n = 1, \ldots, N$. Thus, there are 2^N nonzero maximal minors of $A_{\rm O}$, given by $A_{\rm O}(m_1, \ldots, m_N) = 1$, where $m_n = 2n-1$ or $m_n = 2n$ for $n = 1, \ldots, N$. The N asymptotic line solitons are identified by the index pairs $\{[2n-1, 2n]\}_{n=1}^N$, and the corresponding permutation is $\pi = (2, 1, 4, 3, \ldots, 2N, 2N-1)$ (or $\pi = (12)(34)\cdots(2N, 2N-1)$ in the cycle notation). The amplitude and direction of the nth soliton are $a_n = k_{2n} - k_{2n-1}$ and $c_n = k_{2n-1} + k_{2n}$, respectively. Note that the soliton directions are ordered as $c_1 < c_2 < \cdots < c_N$ due to the ordering of the phase parameters k_n. In fact, the soliton parameters satisfy the inequalities: $c_{n+1} - c_n > a_n + a_{n+1}, n = 1, 2, \ldots, N-1$. Therefore, the asymptotic line solitons for the O-type solutions cannot take arbitrary values of amplitude and direction, and thus do not cover the entire soliton parameter space as was already noted for the two-soliton case in section 3.2. Apart from the position shift of each soliton, the interaction gives rise to a pattern of N intersecting lines in the (x, y)-plane, as shown in figure 7(*a*).

T-type N-soliton solutions. These solutions are obtained by choosing the functions in equation (1.8) as

$$f_n = f^{(n-1)}, \qquad n = 1, \ldots, N \qquad \text{with} \quad f(x, y, t) = \sum_{m=1}^{2N} \mathrm{e}^{\theta_m},$$

which yields the coefficient matrix

$$A_{\rm T} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ k_1 & k_2 & \cdots & k_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ k_1^{N-1} & k_2^{N-1} & \cdots & k_{2N}^{N-1} \end{pmatrix}.$$

In this case all of the $N \times N$ minors of the coefficient matrix A are positive, each being equal to a Van der Monde determinant. These solutions were investigated in [4] where it was shown that they also satisfy the finite Toda lattice hierarchy. The nth line-soliton is labeled by the index pair $[n, n+N]$; that is, its amplitude and direction are determined by the phase parameters (k_n, k_{n+N}). The corresponding permutation is $\pi = (N+1, N+2, \ldots, 2N, 1, 2, \ldots, N)$ (or $\pi = (1, N+1)(2, N+2)\cdots(N, 2N))$ whose open chord diagram has the maximum number of crossings $N(N-1)$. This implies that the T-type N-soliton solution belongs to the top cell of $Gr^+(N, 2N)$. Like the O-type N-soliton solutions, these T-type N-soliton solutions do not cover the whole soliton parameter space. In this case the soliton parameters satisfy the constraints: $|a_{n+1} - a_n| < c_{n+1} - c_n < a_{n+1} + a_n$, $n = 1, 2, \ldots, N - 1$. It was also shown in [4] that these soliton solutions display phenomena of soliton resonance and web structure as shown in figure 7(*b*). Moreover, the intermediate interaction segments are also line solitons because they satisfy the dispersion relation (1.7). All of the asymptotic and intermediate line solitons interact via three-wave resonances. That is, at each interaction vertex or Y-junction (cf figure 1(*c*)), the three interacting line solitons satisfy the resonance condition given by equation (1.9).

P-type N-soliton solutions. Yet another type of N-soliton solutions is obtained by prescribing

$$f_n(x, y, t) = \mathrm{e}^{\theta_n} + (-1)^{N-n}\mathrm{e}^{\theta_{2N-n+1}}, \qquad n = 1, 2, \ldots N,$$

in equation (1.8). The coefficient matrix is given by

$$A_{\mathrm{P}} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & * \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & * & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \iddots & \iddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where the asterisk in the $(2N - n + 1)$th column is equal to $(-1)^{N-n}$. Like A_{O}, the matrix A_{P} also has N pairs of parallel columns labeled by $\{(n, 2N - n + 1), n = 1, \ldots, N\}$ and 2^N non-vanishing minors, and each nonzero minor is 1. The nth soliton is identified by the index pair $[n, 2N - n + 1]$, and the corresponding permutation is $\pi = (2N, 2N - 1, \ldots, 2, 1)$ (or $\pi = (1, 2N)(2, 2N - 1)\cdots(N, N + 1)$). The soliton direction and amplitude are given by $c_n = k_n + k_{2N-n+1}$ and $a_n = k_{2N-n+1} - k_n$, respectively. Note that the soliton directions are not ordered as in the previous two cases. In fact, taking $c_1 = c_2 = \cdots = c_N = 0$ yields the reduction to solutions of the KdV equation [15]. But the soliton amplitudes in this case are ordered as $a_1 > a_2 > \cdots > a_N$. Moreover, the soliton parameters satisfy the constraints: $|c_{n+1} - c_n| < a_n - a_{n+1}$, $n = 1, 2, \ldots, N - 1$. These solutions interact non-resonantly, like the O-type N-solitons, i.e., pairwise with an overall phase shift after collision (see figure 7(*c*)). However, as in the case of the two-soliton solutions in section 3.2, the pairwise phase shifts for P-type solitons is of opposite sign from that of O-type soliton solutions.

4.3. Soliton parameters and pairing map

The most notable differences between the O-, T- and P-type N-soliton solutions in example 4.5 are that they span different regions of the soliton parameter space and that they exhibit dissimilar interaction patterns and phase shifts. However, in addition to these non-resonant and fully resonant solutions, a large family of partially resonant solutions exists when $N > 2$. Thus, the family of N-soliton solutions of KPII is much larger than previously thought, and their classification is indeed nontrivial. Even for $N = 3$, property 4.4(iv) implies that there are 15 inequivalent types of three-soliton solutions. For increasing values of N, it turns out to be a difficult task to classify these solutions according to their coefficient matrices, as was done for $N = 2$ in section 3. Instead, a more direct approach is to enumerate the N-soliton solutions via the involutions $\mathcal{I}_{2N} \subset \mathcal{S}_{2N}$, by constructing a representative coefficient matrix A for each N-soliton equivalence class starting from a pairing map $\pi \in \mathcal{I}_{2N}$. In what follows, we describe a slightly modified approach. We start with the set of amplitudes and directions of the asymptotic line solitons as $|y| \to \infty$. We first recover the soliton pairings from this physical data, then construct the coefficient matrix A from the obtained pairing map. This provides a method to algebraically reconstruct the N-soliton solution unique up to spacetime translations, starting simply from the physical soliton parameters $\{(a_n, c_n)\}_{n=1}^{N}$.

We begin with the following definition of the N-soliton parameter space.

Definition 4.6. *An N-tuple of pairs* $p_N := \{(a_n, c_n)|a_n > 0\}_{n=1}^N \in \mathbb{R}^{2N}$*, of amplitudes and directions for the asymptotic line solitons associated with an N-soliton solution, is defined to be admissible if it yields the set* $\Pi_N := \left\{\left(k_n^-, k_n^+\right) \,\middle|\, k_n^\pm = \frac{1}{2}(c_n \pm a_n)\right\}_{n=1}^N$ *where the* $2N$ *phase parameters* $\left\{k_n^\pm\right\}_{n=1}^N$ *are distinct. The set* $Sol(N)$ *of all admissible N-tuples of amplitudes and directions will be referred to as the N-soliton parameter space.*

Note that the pairs $(a_n, c_n), n = 1, \ldots, N$ in the set p_N are unordered. For example, $p_2 = \left\{(1, 2), \left(\frac{1}{2}, 1\right)\right\}$ and $p_2 = \left\{\left(\frac{1}{2}, 1\right), (1, 2)\right\}$ represent the same two-soliton solution. Similarly, Π_N also consists of unordered pairs $\left(k_n^-, k_n^+\right), n = 1, \ldots, N$. However, since the parameters $\left\{k_n^\pm\right\}_{n=1}^N$ are distinct, they can be sorted in increasing order into an ordered set $K_{2N} = (k_1, k_2, \ldots, k_{2N})$. Hence, Π_N forms a partition of K_{2N} into N distinct pairs. The positions of each pair $\left(k_n^-, k_n^+\right) \in \Pi_N$ can be uniquely identified within K_{2N} by an ordered pair of indices $[i_n, j_n]$ such that $i_n < j_n$. That is, $k_n^- = k_{i_n}$ and $k_n^+ = k_{j_n}$. It is precisely this identification that induces a correspondence between each Π_N and a pairing map $\pi \in \mathcal{I}_{2N} \subset \mathcal{S}_{2N}$, the latter representing a disjoint partition of $[2N]$ into N distinct pairs (see property 4.4(iv)). It is then clear from definition 4.6 that this correspondence also extends between each $p_N \in Sol(N)$ and a $\pi \in \mathcal{I}_{2N}$.

Example 4.7. Consider a three-soliton parameter set $p_3 = \left\{(a_1, c_1) = (1, -3), (a_2, c_2) = \left(\frac{3}{2}, \frac{1}{2}\right), (a_3, c_3) = \left(\frac{3}{2}, \frac{5}{2}\right)\right\}$, and construct the set $\Pi_3 = \left\{\frac{1}{2}(c_n \pm a_n)\right\}_{n=1}^3$. The set p_3 is admissible because the corresponding set $\Pi_3 = \left\{(-2, -1), \left(-\frac{1}{2}, 1\right), \left(\frac{1}{2}, 2\right)\right\}$ contains six distinct phase parameters. Sorting these parameters in increasing order yields $K_6 = \left(-2, -1, -\frac{1}{2}, \frac{1}{2}, 1, 2\right) = (k_1, k_2, k_3, k_4, k_5, k_6)$. Then, $\Pi_3 = \{(k_1, k_2), (k_4, k_6), (k_3, k_5)\}$ which gives the correspondence $p_3 \simeq \Pi_3 \mapsto (12)(35)(46) = \pi$.

Note, however, that distinct elements of $Sol(N)$ associated with solutions in the same equivalence class give rise to identical pairing. In this situation, the corresponding sets Π_N are distinct but after sorting, their elements are ordered in identical fashion into the respective sets K_{2N}. Thus, the soliton parameter space $Sol(N)$ is partitioned into disjoint sectors. Each sector corresponds to an equivalence class of solutions, distinguished by an element $\pi \in \mathcal{I}_{2N}$, equivalently, by the set $\{[e_n, g_n]\}_{n=1}^N$ labeling the N asymptotic line solitons. The total number of such disjoint sectors of $Sol(N)$ equals the cardinality $|\mathcal{I}_{2N}| = (2N-1)!!$.

Once a pairing map $\pi \in \mathcal{I}_{2N}$ is derived from a given N-soliton parameter set p_N, it is then possible to construct a coefficient matrix A satisfying condition 2.2. Clearly, the pivot columns of A will be labeled by the excedance set $\{e_1, \ldots, e_N\}$ of π, and the non-pivot columns are labeled by $\{g_1, \ldots, g_N\}$, where $\pi(e_n) = g_n, n = 1, \ldots, N$. The explicit form of A will be determined by using the rank conditions in proposition 2.6. Recall that in examples 2.7 and 2.8, we demonstrated how to apply the results of proposition 2.6 to a given coefficient matrix A, and obtain the set of index pairs identifying the asymptotic line solitons. In the examples below, we will illustrate the reverse construction. In other words, we will show that the rank conditions of proposition 2.6 are also *sufficient* to construct a coefficient matrix A from a given pairing map π associated with the N-soliton solutions.

Example 4.8. We outline the construction of a coefficient matrix A associated with the three-soliton pairings $\{[1, 2], [3, 5], [4, 6]\}$ found in example 4.7. The construction proceeds in several steps.

Step 1. It follows from property 4.4(ii) that the pivot and non-pivot columns of the 3×6 matrix in RREF are labeled by $\{e_1, e_2, e_3\} = \{1, 3, 4\}$, and $\{g_1, g_2, g_3\} = \{2, 5, 6\}$, respectively. So, the general form of A satisfying condition 2.2 is

$$A = \begin{pmatrix} 1 & z & 0 & 0 & v_1 & w_1 \\ 0 & 0 & 1 & 0 & -v_2 & -w_2 \\ 0 & 0 & 0 & 1 & v_3 & w_3 \end{pmatrix},$$

where $z, \{v_i, w_i\}_{i=1}^{3}$ are non-negative numbers to be determined. Note that each unknown entry can be expressed as certain maximal minors of A (e.g., $A(234) = z$, etc). The negative signs in the second row are included so that the maximal minors of A satisfy the non-negativity condition 2.2(i).

Step 2. In order to obtain further information about A, one needs to apply the rank conditions in proposition 2.6 to the sub-matrices $X[ij]$ and $Y[ij]$ associated with each line soliton $[i, j]$.

If we start with the line-soliton $[1, 2]$, and consider the sub-matrix $Y[12] = \emptyset$, we find that $\mathrm{rank}(Y[12]) = 0$. Then according to proposition 2.6(ii), $\mathrm{rank}(Y[12]|1, 2) = 1$, so that columns 1 and 2 are proportional. Hence, $z \neq 0$. The sub-matrix $X[12]$ consists of columns $3, \ldots, 6$ of A above. From proposition 2.6(i), it follows that $\mathrm{rank}(X[12]) \leqslant N - 1 = 2$, since $N = 3$. Then any maximal minor of A consisting of the columns of $X[12]$ will vanish. In particular, we find that $A(345) = v_1 = 0$ and $A(346) = w_1 = 0$, so that

$$A = \begin{pmatrix} 1 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -v_2 & -w_2 \\ 0 & 0 & 0 & 1 & v_3 & w_3 \end{pmatrix}.$$

Step 3. Next we consider the index pair $[3, 5]$. As the sub-matrix $Y[35] = (0, 0, 1)^T$ is of rank 1, we obtain $\mathrm{rank}(Y[35]|5) = 2$ from proposition 2.6(ii). Therefore, columns 4 and 5 of A are linearly independent. Then $v_2 \neq 0$. The sub-matrix $X[35]$ consisting of the columns 1, 2 and 6 of A, is of rank 2, because its column space is spanned by the linearly independent columns $(1, 0, 0)^T$ and $(0, -w_2, w_3)^T$. Then from proposition 2.6(i), $\mathrm{rank}(X[35]|3) = 3 \Rightarrow A(136) = w_3 \neq 0$, and $\mathrm{rank}(X[35]|5) = 3 \Rightarrow A(156) = v_3 w_2 - v_2 w_3 \neq 0$. Moreover, the non-negativity of all maximal minors of A requires that z, v_2, w_3 and $A(156)$ must be positive. In particular, $A(156) = v_3 w_2 - v_2 w_3 > 0$ implies that $v_3 \neq 0$, $w_2 \neq 0$, so these are also positive. Thus, the matrix A is parametrized by five positive parameters which can be chosen as follows: $t_1 = z, t_2 = v_2, t_3 = v_3, t_4 = w_3/v_3, t_5 = w_2/v_2 - w_3/v_3$. It can be directly verified that all nonzero maximal minors of A are polynomials in $t_1, \ldots, t_5$ with positive coefficients.

Thus, we have constructed a five-parameter family of coefficient matrix A that will generate the asymptotic line solitons $[1, 2], [3, 5], [4, 6]$ for any choice of the positive parameter values $t_1, \ldots, t_5$, via proposition 2.6. Note that the vanishing minors of A satisfy the duality conditions of equation (4.5). Furthermore, any such A together with the phase parameters in the set K_6 of example 4.7 would generate a three-soliton solution $u(x, y, t)$ with soliton parameters p_3. This solution is unique up to spacetime translations corresponding to different choices for the phase constants $\theta_{m,0}$, $m = 1, \ldots, 6$ in equation (1.5).

We make a few observations from the above example. First, note that the rank conditions together with non-negativity of the maximal minors completely determine which maximal minors of A are zero, and which are nonzero. Second, the non-vanishing maximal minors are chosen to be positive by expressing them in terms of a suitable set of freely prescribed positive parameters which are rational expressions in the matrix elements of A. This parametrization completely determines the coefficient matrix A which is in RREF because all the entries in its non-pivot columns can be expressed as appropriate maximal minors. Third, both rank conditions (propositions 2.6(i) and (ii)) are applied to the same index pair $[e_n, g_n]$ since it labels a pair of asymptotic line solitons as $|y| \to \infty$. Instead, one could also apply the rank conditions from either proposition 2.6(i) or (ii) to each index pair, and recover the remaining information from the duality condition equation (4.5). For example, since columns 1 and 2 are proportional for the coefficient matrix A of example 4.8, the minors $A(125) = A(126) = 0$. Then the duality condition implies that $A(346) = A(345) = 0$ as we found in step 2. We also point out here that in [15] the N-soliton τ-function was required to satisfy a set of conditions, which was referred to as the 'N-soliton condition' (definition 4.2 of [15]). These conditions follow directly from the rank and duality conditions discussed in this paper. We reiterate the above observations through another example.

Example 4.9. In this example, we construct a four-soliton solution starting with the following set of soliton parameters: $p_4 = (a_n, c_n)_{n=1}^4 = \left\{(1, -3), \left(\frac{3}{2}, -\frac{3}{2}\right), \left(\frac{3}{2}, \frac{1}{2}\right), (1, 2)\right\} \in Sol(4)$.

Step 1. First, we construct the set $\Pi_4 = \left\{(-2, -1), \left(-\frac{3}{2}, 0\right), \left(-\frac{1}{2}, 1\right), \left(\frac{1}{2}, \frac{3}{2}\right)\right\}$ from p_4 using $k_n^{\pm} = \frac{1}{2}(c_n \pm a_n)$. Then we obtain the ordered set $K_8 = \left(-2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}\right) = (k_1, \ldots, k_8)$ by sorting the phase parameters in Π_4 which can now be re-expressed as $\Pi_4 = \{(k_1, k_3), (k_2, k_5), (k_4, k_7), (k_6, k_8)\}$. This gives the correspondence $\Pi_4 \mapsto (13)(25)(47)(68) = \pi \in \mathcal{I}_8$, where π is expressed as products of disjoint 2-cycles. The asymptotic line solitons are identified by the set $\{[1, 3], [2, 5], [4, 7], [6, 8]\}$ of index pairs.

Step 2. We proceed to construct the coefficient matrix A that will generate the τ-function of the four-soliton solution. As in the previous example, we start with the 4×8 matrix A satisfying condition 2.2:

$$A = \begin{pmatrix} 1 & 0 & -z_1 & 0 & u_1 & 0 & -v_1 & -w_1 \\ 0 & 1 & z_2 & 0 & -u_2 & 0 & v_2 & w_2 \\ 0 & 0 & 0 & 1 & u_3 & 0 & -v_3 & -w_3 \\ 0 & 0 & 0 & 0 & 0 & 1 & v_4 & w_4 \end{pmatrix},$$

whose pivots and non-pivot indices are $\{e_1, e_2, e_3, e_4\} = \{1, 2, 4, 6\}$, and $\{g_1, g_2, g_3, g_4\} = \{3, 5, 7, 8\}$, respectively, and where z_1, z_2, u_1, u_2, u_3 and $v_i, w_i, i = 1, \ldots, 4$ are non-negative reals to be determined.

Step 3. We then apply the rank conditions from proposition 2.6 systematically to each soliton index pair $[i, j]$, and collect all the information regarding the unknown entries in terms of zero and nonzero minors of A. Here we indicate only the essential steps.

Consider the index pair $[1, 3]$ and the associated sub-matrix $X[13]$ which consists of columns $4, \ldots, 8$ of A above. Since $\text{rank}(X[13]) \leqslant N - 1 = 3$, the maximal minors: $A(l_1, \ldots, l_4) = 0, \{l_1, \ldots, l_4\} \subset \{4, 5, 6, 7, 8\}$. For the index pair $[6, 8]$, the sub-matrix $Y[68] = (-v_1, v_2, -v_3, v_4)^T$ is clearly of rank 1. Then from proposition 2.6(ii) $\text{rank}(Y[68]|6, 8) = 2$, which implies that $A(l, 6, 7, 8) = 0$ for any $l \in [8]$. Putting these conditions together, it can be shown that all vanishing minors of A obtained via proposition 2.6 are generated by the following relations:

$$u_i v_j - u_j v_i = 0, \qquad u_i w_j - u_j w_i = 0, i, j = 1, 2, \qquad v_i w_j - v_j w_i = 0, i, j = 1, 2, 3. \tag{4.6}$$

Next we determine the nonzero entries of A. Note that $z_1 \neq 0$ since columns 2 and 3 of A are linearly independent. This is because the sub-matrix $Y[13] = (0, 1, 0, 0)^T$ associated with the index pair $[1, 3]$, is of rank 1, then proposition 2.6(ii) implies that $\text{rank}(Y[13]|3) = 2$. Consider now the sub-matrix $Y[47]$ whose rank is 2 since it is spanned by the linearly

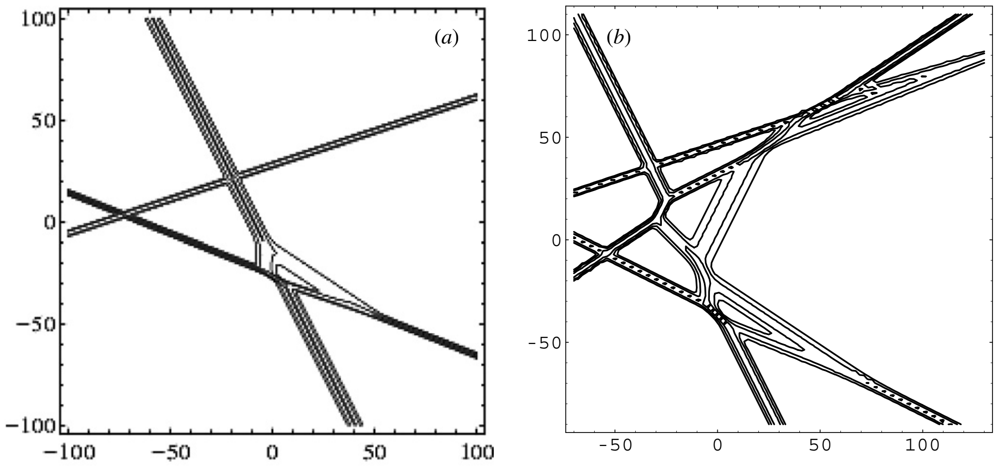

Figure 8. (*a*) A three-soliton solution of example 4.8 at $t = 12$, generated by the coefficient matrix A with parameter values: $z = v_2 = w_2 = 1, v_3 = 3, w_3 = \frac{5}{3}$; (*b*) a four-soliton solution of example 4.9 at $t = 20$. The parameters of the coefficient matrix A are given by $z_1 = u_1 = u_2 = v_1 = v_2 = v_3 = 1, z_2 = u_3 = v_4 = w_1 = w_2 = w_3 = 2, w_4 = 3$.

independent columns 5 and 6 of A. Then columns 4, 5 and 6 are also linearly independent as $\mathrm{rank}(Y[47]|4) = 3$. Moreover, these columns also span the column space of the sub-matrix $X[13]$ and form a basis which we denote by β_{13}. By similar reasoning, we find the basis sets β_{25} and β_{47} for the column spaces of $X[25]$ and $X[47]$, respectively. These are given by

$$\beta_{13} = \begin{pmatrix} 0 & u_1 & 0 \\ 0 & -u_2 & 0 \\ 1 & u_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \beta_{25} = \begin{pmatrix} 1 & 0 & -v_1 \\ 0 & 0 & v_2 \\ 0 & 0 & -u_3 \\ 0 & 1 & v_4 \end{pmatrix}, \qquad \beta_{47} = \begin{pmatrix} 1 & 0 & -w_1 \\ 0 & 1 & w_2 \\ 0 & 0 & -w_3 \\ 0 & 0 & w_4 \end{pmatrix}.$$

Since both $\mathrm{rank}(X[13]|1) = \mathrm{rank}(X[13]|3) = 4$, using the basis β_{13} we get $A(1456) = u_2 > 0$, and $A(3456) = z_2u_1 - z_1u_2 > 0$ where we also required that the nonzero maximal minors to be positive. In addition, if we take $z_1 > 0$ (since $z_1 \neq 0$ from above), then $z_2 > 0, u_1 > 0$, as well. Proceeding in a similar fashion with β_{25} and β_{47} we find that that $u_3, v_2, v_3, v_4, w_3, w_4$ are all positive. Then from equation (4.6), we finally conclude that all entries in the non-pivot columns of A are nonzero unlike the previous example. However, these nonzero elements must satisfy the three independent constraints arising from equation (4.6). Furthermore, by direct computation using the matrix elements of A we find that all vanishing minors of A are in fact generated by the relations in equation (4.6) which arise from the rank conditions of proposition 2.6.

As in example 4.8, it is possible to construct a set of positive parameters such that all nonzero maximal minors of A are polynomials in these parameters with positive coefficients. These are given by

$$t_1 = z_1, \qquad t_2 = z_2, \qquad t_3 = \frac{u_1}{z_1} - \frac{u_2}{z_2}, \qquad t_4 = \frac{u_2}{z_2}, \qquad t_5 = u_3,$$
$$t_6 = \frac{v_2}{u_2} - \frac{v_3}{u_3}, \qquad t_7 = \frac{v_3}{u_3}, \qquad t_8 = v_4, \qquad t_9 = \frac{w_4}{v_4} - \frac{w_3}{u_3}, \qquad t_{10} = \frac{w_4}{v_4}.$$

Figure 8(*a*) shows a three-soliton solution with soliton parameters given in example 4.7 and a coefficient matrix A constructed in example 4.8. Figure 8(*b*) shows a four-soliton solution with soliton parameters and matrix A from example 4.9. It should be noted that the above examples are only illustrations of the general result asserting that for every pairing map $\pi \in \mathcal{I}_{2N}$ there exists a parametrized family of $N \times 2N$ matrices A which satisfy condition 2.2 and generate an equivalence class of N-soliton solutions. The proof of this remarkable result will be given in a future work [7].

4.4. Combinatorics of N-soliton solutions

Further refinement of the N-soliton classification scheme can be achieved by studying the combinatorial properties of the associated $N \times 2N$ coefficient matrix A. We have already shown that the N-soliton solution space is characterized by the set $\mathcal{I}_{2N}$ of fixed point free involutions of the permutation group $\mathcal{S}_{2N}$. In turn, these involutions of $\mathcal{S}_{2N}$ can be enumerated in terms of the various possible arrangements of the pivot and non-pivot columns of the N-soliton coefficient matrix A. A more geometric classification of the N-soliton solutions using Schubert decomposition of the real Grassmannian $Gr(N, 2N)$ has been carried out in [15] where the arrangements of pivot and non-pivot indices were described in terms of Young diagrams. We resort to a more elementary treatment here and present the main results below.

Proposition 4.10. *Suppose that the index sets* $\{e_1, \ldots, e_N\}$ *and* $\{g_1, \ldots, g_N\}$ *with* $e_n < g_n, n = 1, \ldots, N$*, form a disjoint partition of* $[2N]$*, and label respectively, the pivot and non-pivot columns of a coefficient matrix A associated with a N-soliton solution of KPII. Then, the following results hold.*

(i) The index set $\{e_1, \ldots, e_N\}$ *is ordered as follows:* $1 = e_1 < e_2 < \cdots < e_N < 2N$*. Moreover, the elements satisfy* $n \leqslant e_n \leqslant 2n - 1, n = 1, \ldots, N$*.*

(ii) The total number of choices C_N *for the ordered set* $\{e_1, \ldots, e_N\}$ *in item (i) is given by the Nth Catalan number (see, e.g. [27]),*

$$C_N = \frac{(2N)!}{N!(N+1)!}. \tag{4.7}$$

(iii) The set $\{g_1, \ldots, g_N\}$ *is unordered, and the element* g_n *can be chosen in* $2n - e_n$ *ways for* $n = 1, \ldots, N$*. Therefore, the number of possible choices for the unordered set* $\{g_1, \ldots, g_N\}$ *for each set* $\{e_1, \ldots, e_N\}$ *is given by*

$$m(e_1, \ldots, e_N) = \prod_{n=1}^{N} (2n - e_n). \tag{4.8}$$

It follows from proposition 4.10 that the total number of distinct equivalence classes of N-soliton solutions satisfies the curious combinatorial identity

$$F_N := \sum_{\substack{e_1 < \cdots < e_N, \\ n \leqslant e_n \leqslant 2n-1}} m(e_1, \ldots, e_N) = (2N-1)!!, \tag{4.9}$$

which can be proved by following a similar line of argument that is provided below in the proof of proposition 4.10(ii). Items (i) and (iii) of proposition 4.10 were already proved in [15].

Proof. (*Proposition 4.10(ii)*) Let E_N denote the set of all N-tuples $(e_1, \ldots, e_N)$ for which proposition 4.10(i) holds. Then it is clear that $|E_N| = C_N$. Since $e_1 = 1$, each N-tuple contains one or more indices satisfying $e_m = 2m - 1, m = 1, \ldots, N$. Then by sorting the elements of E_N according to the *largest* positive integer $n \in [N]$ such that $e_n = 2n-1$, we obtain the disjoint partition

$$E_N = \bigsqcup_{n=1}^{N} W_n, \qquad \text{where} \quad W_n = \{(e_1,\ldots,e_N)|e_n = 2n-1, e_m < 2m-1, m > n\}.$$

Note that W_n can be expressed as the direct product: $W_n = E_{n-1} \times \{e_n = 2n-1\} \times \widehat{E}_{N-n}$, where the set $E_{n-1} = \{(e_1,\ldots,e_{n-1})\}$ with $j \leqslant e_j \leqslant 2j-1$, $j \in [n-1]$, is defined similarly as E_N, and where $\widehat{E}_{N-n} = \{(e_{n+1},\ldots,e_N)\}$ with $2n+j-1 \leqslant e_{n+j} < 2(n+j)-1$, $j \in [N-n]$. If we define new indices $\widehat{e}_j := e_{n+j} - (2n-1)$, $j = 1,\ldots,N-n$ and relabel the elements of $\widehat{E}_{N-n}$, then it should be clear that

$$\widehat{E}_{N-n} \simeq \{(\widehat{e}_1,\ldots,\widehat{e}_j)|j \leqslant \widehat{e}_j \leqslant 2j-1\} =: E_{N-n}.$$

Now the cardinalities of E_{n-1} and $\widehat{E}_{N-n}$ are C_{n-1} and C_{N-n}, respectively. Then it follows from above that $|W_n| = C_{n-1}C_{N-n}$, and

$$|E_N| = C_N = \sum_{n=1}^{N} C_{n-1}C_{N-n}. \tag{4.10}$$

Equation (4.10) is the well known recursion relation for the Catalan numbers [5, 27] C_N, $N \geqslant 1$ with $C_0 := 1$. If $C(z) = \sum_{N=0}^{\infty} C_N z^N$ is the generating function of the C_N, then equation (4.10) implies that $C(z)$ satisfies $zC^2(z) - C(z) + 1 = 0$. This yields

$$C(z) = \frac{1-\sqrt{1-4z}}{2z} = \sum_{N=0}^{\infty} \frac{(2N)!}{N!(N+1)!} z^N,$$

by choosing the root consistent with $C(0) = C_0 = 1$, then expanding it in power series. Finally, we obtain the desired result by equating the coefficients of the two power series for $C(z)$. □

In view of proposition 4.10(i), it is natural to associate with each ordered set $\{e_1,\ldots,e_N\}$ a *weight vector* w and its length $\|w\|$ defined by

$$w := (w_1, w_2, \ldots, w_N), \qquad \text{where} \quad w_n := e_n - n \geqslant 0, \quad n \in [N],$$
$$\text{and} \qquad \|w\| := \sum_{n=1}^{N} (e_n - n), \tag{4.11}$$

respectively. Note that the weights form a non-decreasing sequence: $0 = w_1 \leqslant w_2 \leqslant \cdots \leqslant w_N$ and $0 \leqslant w_n \leqslant n-1$. Similarly, we associate the unordered set of non-pivot indices $\{g_1,\ldots,g_N\}$ with an *inversion* vector σ defined by

$$\sigma := (\sigma_1, \sigma_2 \ldots \sigma_N), \qquad \text{where} \quad \sigma_n := |\{g_j|g_j > g_n, j < n\}|, \qquad n \in [N]. \tag{4.12}$$

The inversions satisfy $0 \leqslant \sigma_n \leqslant 2n - e_n - 1$, $n = 1,\ldots,N$. The upper limit of σ_n follows from proposition 4.10(iii), by placing g_n to the leftmost of the $2n - e_n$ available positions, and filling the remaining positions with indices g_j such that $j < n$. Note that the pair of vectors (w, σ) are related to the pair (Y^+, Y^-) of Young diagrams introduced in [15].

The results of proposition 4.10 together with the weight and inversion vectors provide a refinement of the classification scheme for the N-soliton solutions. We illustrate here the refined scheme for $N = 3$. In this case, A is a 3×6 matrix with three pivots satisfying $e_1 = 1, 2 \leqslant e_2 \leqslant 3, 3 \leqslant e_3 \leqslant 5$. From proposition 4.10(ii), the total number of pivot configurations $\{e_1, e_2, e_3\}$ is given by the Catalan number $C_3 = 5$. Thus, there are five subclasses of three-soliton solutions depending on five distinct pivot configurations which

Table 1. Weight vectors and possible pivot arrangements for three-soliton solutions.

$\|w\|$	w	$\{e_1, e_2, e_3\}$
0	(0, 0, 0)	{1, 2, 3}
1	(0, 0, 1)	{1, 2, 4}
2	(0, 0, 2)	{1, 2, 5}
	(0, 1, 1)	{1, 3, 4}
3	(0, 1, 2)	{1, 3, 5}

Table 2. The 15 distinct three-soliton solutions.

$\{e_1, e_2, e_3\}$	$\{g_1, g_2, g_3\}$	σ	Three-soliton solution
{1, 2, 3}	{4, 5, 6}	(0, 0, 0)	{[1, 4], [2, 5], [3, 6]}
	{4, 6, 5}	(0, 0, 1)	{[1, 4], [2, 6], [3, 5]}
	{5, 4, 6}	(0, 1, 0)	{[1, 5], [2, 4], [3, 6]}
	{5, 6, 4}	(0, 0, 2)	{[1, 5], [2, 6], [3, 4]}
	{6, 4, 5}	(0, 1, 1)	{[1, 6], [2, 4], [3, 5]}
	{6, 5, 4}	(0, 1, 2)	{[1, 6], [2, 5], [3, 4]}
{1, 2, 4}	{3, 5, 6}	(0, 0, 0)	{[1, 3], [2, 5], [4, 6]}
	{3, 6, 5}	(0, 0, 1)	{[1, 3], [2, 6], [4, 5]}
	{5, 3, 6}	(0, 1, 0)	{[1, 5], [2, 3], [4, 6]}
	{6, 3, 5}	(0, 1, 1)	{[1, 6], [2, 3], [4, 5]}
{1, 2, 5}	{3, 4, 6}	(0, 0, 0)	{[1, 3], [2, 4], [5, 6]}
	{4, 3, 6}	(0, 1, 0)	{[1, 4], [2, 3], [5, 6]}
{1, 3, 4}	{2, 5, 6}	(0, 0, 0)	{[1, 2], [3, 5], [4, 6]}
	{2, 6, 5}	(0, 0, 1)	{[1, 2], [3, 6], [4, 5]}
{1, 3, 5}	{2, 4, 6}	(0, 0, 0)	{[1, 2], [3, 4], [5, 6]}

are determined by the associated weight vectors. These subclasses are listed in table 1 in increasing order of the length $\|w\|$ of the weight vector. For each of these pivot arrangements, proposition 4.10(iii) gives the total number of distinct non-pivot configurations. Each set $\{g_1, g_2, g_3\}$ in a given subclass is distinguished by its unique inversion vector. The various three-soliton solutions in each subclass are shown in table 2, and the associated chord diagrams are presented in figure 9. Note that instead of open chord diagrams we use circles in figure 9. Here each straight chord replaces the pair of upper and lower chords connecting the same index pair $[e_n, g_n]$ in the self-dual, open chord diagrams of the *N*-soliton solutions (cf property 4.4(iv)).

The subclass associated with the pivot configuration $\{1, 2, 3\}$ is isomorphic to the permutation group $\mathcal{S}_3$ acting on $\{4, 5, 6\}$ to form the non-pivot sets $\{g_1, g_2, g_3\}$. These are arranged in the second column of table 2 according to the non-decreasing order of the L^1-norm $|\sigma| = \sigma_1 + \sigma_2 + \sigma_3$ of the respective inversion vectors in the second column of table 2. The corresponding set of diagrams forming a hexagon in figure 9 is the permutahedron for $\mathcal{S}_3$. On the other hand, when $\{e_1, e_2, e_3\} = \{1, 2, 4\}$ the non-pivot index g_3 is chosen in $2 \times 3 - 4 = 2$ ways; g_2 is chosen in $2 \times 2 - 2 = 2$ ways; and obviously there is only one way to choose g_1. Thus, there are only 4 (instead of 6) possible ways the non-pivot columns 3, 5 and 6 can be arranged to form the set $\{g_1, g_2, g_3\}$. This is due to the restriction that $g_3 \neq 3$ because according to proposition 4.10, g_3 must be greater than $e_3 = 4$. The chord diagrams with the pivot set $\{1, 2, 4\}$ form a square which is a subpolytope of the permutahedron of $\mathcal{S}_3$. We will discuss the polytope structure for the *N*-soliton solutions in [7].

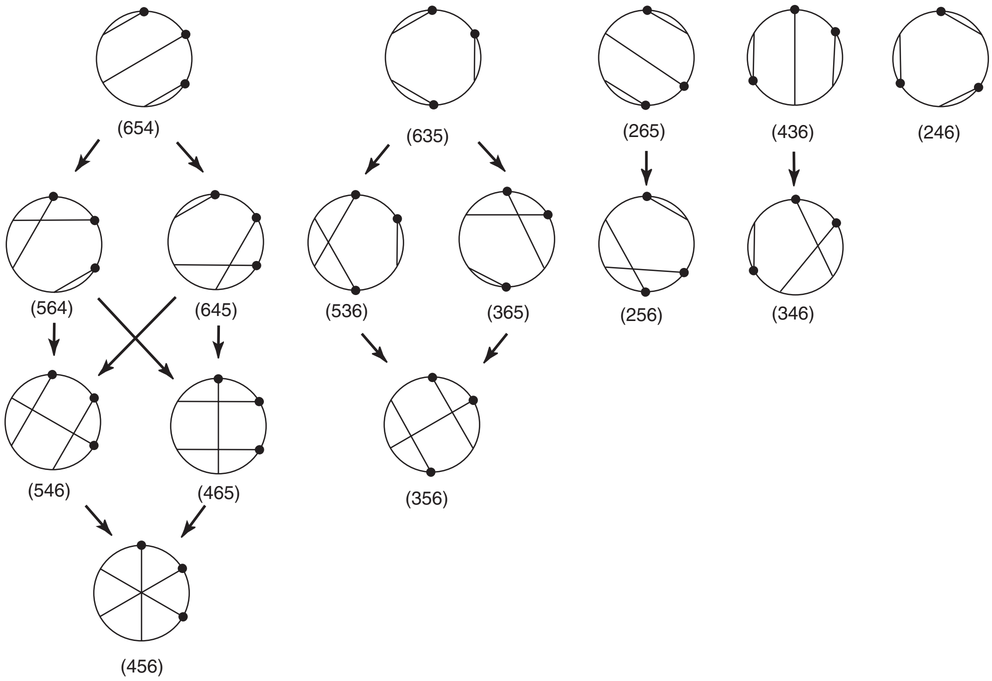

Figure 9. The circular chord diagrams for three-soliton solutions. On each circle, the indices are increasing clockwise from the top. The dots on each diagram indicate the pivots (e_1, e_2, e_3), and the ordered numbers below the diagrams indicate (g_1, g_2, g_3). The number of the diagrams having the same number of crossings is given by the generating function $F_3(q) = q^3+3q^2+6q+5$, where 5 is the Catalan number $C_3 = F_3(0)$.

With the circular chord diagrams for the sets $\{e_1, \dots, e_N\}$ and $\{g_1, \dots, g_N\}$, one can find a q-analog of the function $m(e_1, \dots, e_N)$ in equation (4.8) defined by

$$m(e_1, \dots, e_N)(q) = \prod_{n=1}^{N}[2n - e_n]_q = \sum_{c=0}^{c_{\max}} m_c q^c.$$

Here m_c gives the number of the circular chord diagram having c crossings, and the maximum number of crossings for given $\{e_1, \dots, e_N\}$ is [7]

$$c_{\max} = N^2 - \sum_{n=1}^{N} e_n.$$

For example, when $\{e_1, e_2, e_3\} = \{1, 2, 3\}$ and $\{e_1, e_2, e_3\} = \{1, 2, 4\}$, we have

$$m(1, 2, 3)(q) = q^3 + 2q^2 + 2q + 1 \qquad \text{and} \qquad m(1, 2, 4)(q) = q^2 + 2q + 1,$$

corresponding to the hexagon and square in figure 9. Note that $m(e_1, \dots, e_N)(q = 1) = m(e_1, \dots, e_N)$. One can also define a q-analog of the function F_N in equation (4.9) which gives the number of circular chord diagrams having c crossings for given N, i.e. the number of N-soliton solutions having c T-type pairwise interactions among the N asymptotic line solitons:

$$F_N(q) := \sum_{\{e_1,\dots,e_N\}} m(e_1, \dots, e_N)(q).$$

For example, we have $F_3(q) = q^3 + 3q^2 + 6q + 5$ as is evident from counting the number of circular chord diagrams (from bottom to top) at each crossing level in figure 9. Note also that $F_N(1) = F_N$ and $F_N(0) = C_N$, the Nth Catalan number which equals the number of possible pivot configurations $[\{e_1, \dots, e_N\}]$, as well as the number of chord diagrams with no crossings. The latter case represented by the diagrams in the top row of figure 9 yields the number of N-soliton solutions with only P- and O-type interactions among the N line solitons. Furthermore, if we define the generating function by

$$F(q,x) := \sum_{N=0}^{\infty} F_N(q) x^N,$$

then it is possible to show that $F(q,x)$ can be expressed by the continued fraction of the Stieltjes type,

$$F(q,x) = \cfrac{1}{1 - \cfrac{x[1]_q}{1 - \cfrac{x[2]_q}{1 - \cfrac{x[3]_q}{1 - \cdots}}}},$$

with $F_0(q) = 1$. A proof can be found in [6] (see also [14]).

Acknowledgments

SC thanks Gino Biondini for useful discussions and some figures. This work was partially supported by the National Science Foundation Grant nos DMS-0307181 and DMS-0404931.

References

[1] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)

[2] Biondini G and Chakravarty S 2006 Soliton solutions of the Kadomtsev–Petviashvili II equation *J. Math. Phys.* **47** 033514

[3] Biondini G and Chakravarty S 2007 Elastic and inelastic line-soliton solutions of the Kadomtsev–Petviashvili II equation *Math. Comput. Simul.* **74** 237–50

[4] Biondini G and Kodama Y 2003 On a family of solutions of the Kadomtsev–Petviashvili equation which also satisfy the Toda lattice hierarchy *J. Phys. A: Math. Gen.* **36** 10519–36

[5] Bóna M 2004 *Combinatorics of Permutations* (Boca Raton, FL: Chapman and Hall)

[6] Chakravarty S and Kodama Y 2008 A generating function for the N-soliton solutions of the Kadomtsev–Petviashvili II equation *Contemp. Math.* at press (*Preprint* arXiv:0802.0524)

[7] Chakravarty S and Kodama Y Combinatorics and geometry of the N-soliton solutions of the KP equation, in preparation

[8] Corteel S 2007 Crossings and alignments of permutations *Adv. Appl. Math.* **38** 149–63

[9] Freeman N C 1980 Soliton interactions in two dimensions *Adv. Appl. Mech.* **20** 1–37

[10] Freeman N C and Nimmo J J C 1983 Soliton-solutions of the Korteweg–de Vries and Kadomtsev–Petviashvili equations: the Wronskian technique *Phys. Lett.* **95A** 1–3

[11] Hirota R 2004 *The Direct Method in Soliton Theory* (Cambridge: Cambridge University Press)

[12] Infeld E and Rowlands G 2000 *Nonlinear Waves, Solitons and Chaos* (Cambridge: Cambridge University Press)

[13] Kadomtsev B B and Petviashvili V I 1970 On the stability of solitary waves in weakly dispersing media *Sov. Phys.—Dokl.* **15** 539–41

[14] Kasraoui A and Zeng J 2006 Distribution of crossings, nestings and alignments of two edges in matchings and partitions *Elect. J. Comb.* **13** R33, 1–12

[15] Kodama Y 2004 Young diagrams and N-soliton solutions of the KP equation *J. Phys. A: Math. Gen.* **37** 11169–90

[16] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)

[17] Medina E 2002 An *N* soliton resonance for the KP equation: interaction with change of form and velocity *Lett. Math. Phys.* **62** 91–9

[18] Miles J W 1977 Diffraction of solitary waves *J. Fluid Mech.* **79** 171–9

[19] Miwa T, Jimbo M and Date E 2000 *Solitons: Differential Equations, Symmetries and Infinite-dimensional Algebras* (Cambridge: Cambridge University Press)

[20] Newell A C and Redekopp L 1977 Breakdown of Zakharov–Shabat theory and soliton creation *Phys. Rev. Lett.* **38** 377–80

[21] Novikov S P, Manakov S V, Pitaevskii L P and Zakharov V E 1984 *Theory of Solitons. The Inverse Scattering Transform* (New York: Plenum)

[22] Ohkuma K and Wadati M 1983 The Kadomtsev–Petviashvili equation: the trace method and the soliton resonances *J. Phys. Soc. Japan* **52** 749–60

[23] Pashaev O and Francisco M 2005 Degenerate four virtual soliton resonance for the KP-II *Theor. Math. Phys.* **144** 1022–9

[24] Postnikov A 2006 Total positivity, Grassmannians, and networks *Preprint* math.CO/0609764

[25] Roselle D P 1968 Permutations by number of rises and successions *Proc. Am. Math. Soc.* **19** 8–16

[26] Sato M 1981 Soliton equations as dynamical systems on an infinite dimensional Grassmannian manifold *RIMS Kokyuroku* **439** 30–46

[27] Stanley R P 1997 *Enumerative Combinatorics (Cambridge Studies in Advanced Mathematics)* vol 2 (Cambridge: Cambridge University Press)

[28] Williams L K 2005 Enumeration of totally positive Grassmann cells *Adv. Math.* **190** 319–42